Universidade Federal de Minas Gerais
Instituto de Ciências Exatas
Departamento de Matemática

Localization formulas on complex supermanifolds

by

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Comissão Examinadora:

Dr. Maurício Barros Correa Júnior - Orientador

Dr. Miguel Rodríguez Peña - Coorientador

Dr. Márcio Gomes Soares

Dr. Marcos Jardim

Dr. Alejandro Cabrera

Dr. Arturo Ulises Fernandez Perez

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"Saber para poder, poder para agir, agir para ser."

Olívio Montenegro

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Contents

Resumo

Nesta tese estudamos uma teoria de localização para super campos vetoriais holomorfos em supermanifolds complexas e compactas. Mostramos teoremas de resíduos para super campos vetoriais holomorfos pares e ímpares, com singularidades isoladas e não-degeneradas, e determinamos os resíduos sob certas condições locais.

Abstract

In this Thesis we will study a localization theory for even and odd holomorphic super vector fields on compact complex supermanifolds. We prove residue theorems for vector fields with non-degenerated and isolated singularities. Moreover, we determine the residues under certain local conditions.

Introduction

The calculation of certain integrals is very important in mathematics and Physics, especially in String Theory. Localization techniques has been used by theoretical physicists in order to calculate integrals with the purpose to determinate partition functions, see [26] and the survey [15] about applications of localization techniques to supersymmetric quantum field theories.

In [10] Duistermaat and Heckman provided a localization formula for a torus action on a symplectic manifold. Later independently in [4] Berline and Vergne and in [1] Atiyah and Bott generalized the Duistermaat-Heckman formula. This formula is known as The Berline-Vergne-Atiyah-Bott localization formula. In [25] Witten has provided a proof of this result by using supergeometry techniques.

More precisely, the Duistermaat-Heckman Formula can be stated as follows: Let V be a vector field, with only nondegenerate zero isolated zeros, on a symplectic manifold (M,ω) of real dimension $2n$, with $(d+i_V)\omega = 0$ and such that there is a smooth function g such that $i_V\omega = dg$, then for any $s>0$,

$$\int_X \exp\{\omega - sg\} = \int_X e^{-sg}\frac{\omega^n}{n!} = \left(\frac{2\pi}{i}\right)^n \sum_{p_\kappa \in Sing(V)} \left[\frac{e^{-sg(p_\kappa)}}{(is)^n \det^{\frac{1}{2}}(JV(p_\kappa))}\right](p_\kappa)\,,$$

where $p_\kappa \in Sing(V)$ is a singular point of V and $JV(p_\kappa)$ denotes the jacobian of V at p_κ.

It is worth mentioning that Chern-Gauss-Bonnet and Poincaré-Hopf Theorems were the first results in which the localization phenomena appear. Bott in [6] generalized this theorem and in [3] Baum and Bott provided a localization theory for holomorphic foliations. More precisely, let X be a compact complex manifold v is a holomorphic vector field with isolated singularities on X. The Poincaré-Hopf Theorem and Chern-

Gauss-Bonnet Theorem imply that

$$\int_X c_n(TX) = \sum_{p\in\{v(p)=0\}} Ind_p(v),$$

where X is a compact complex manifold, $c_n(TX)$ denotes its top Chern class and $Ind_p(v)$ is the Poincaré-Hopf index of v on p. In [22] Weiping gave a simple proof of the Bott residue theorem in a slightly more general form by using Witten and Bismut [5] techniques.

A complex supermanifold is a ringed space $S = (X, \mathcal{O}_S)$, where X is a complex manifold and the sheaf of rings $\mathcal{O}_S$ on X is locally isomorphic to an exterior algebra over a vector bundle. Let $TS := Der(\mathcal{O}_S)$ be the tangent bundle of $S = (X, \mathcal{O}_S)$ and v a holomorphic vetor field on X. Let $J(v)$ be the jacobian of v. Suppose that the singular set $Sing(v) = \{v = 0\}$ is isolated and non-degenerate. That is, the superdeterminant([24]) satisfies $\mathrm{Ber}(V)(p_j) \neq 0$ for all $p \in Sing(v)$.

In the real supermanifold context Schwarz and Zaboronsky in [18] have provided a localization formula for odd supervector fields. See [7] for a similar result due to Bruzzo and Fucito and [27] for a Zakharevich's result for odd vector fields with non-isolated singularities.

In this Thesis, we prove, for even and odd holomorphic vector fields, the following results:

Theorem 0.1. Let S a complex supermanifold of dimension $n|m$. Let V be an even or odd (with $n = m$) supervector without singularities, then for any $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, we have:

$$\int_S \eta = 0.$$

Theorem 0.2. Let V be an even or odd (with $n = m$) holomorphic supervector field on S with isolated singularities $p_i \in Sing(V)$, then for any $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, we have:

$$\int_S \eta = \sum_i Res_{p_i}(V, \eta)$$

where

$$Res_{p_i}(V, \eta) = \lim_{t\to 0} \int_{SB_\epsilon(p_i)} \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t} \right\}$$

For the even case, we prove the following residue formula:

Theorem 0.3. Let V be an even holomorphic vector field with a non-degenerate isolated singularity $p_j \in Sing(V)$, let S be a compact complex supermanifold with dimension $m|n$

and let $\eta \in \oplus A^{(p,q)|(r,s)}$ be such that $(\overline{\partial} + i_V)(\eta) = 0$. If $\det(\mathbb{B}(D))(\mathbb{B}(p_j)) = 1$, then we have:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^m \frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{\mathrm{Ber}(V)}(p_j)$$

For the odd case, we prove the following residue formulas:

Theorem 0.4. Let S be a compact complex supermanifold of $n|n$ dimension and let V be an odd holomorphic vector field with a non-degenerate isolated singularity $p_j \in Sing(V)$, whose representation in local coordinates is equal to $V = \sum_{i=1}^n f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i \frac{\partial}{\partial \xi_i}$, where $g_i(z)$ are even functions without odd variables and $f_i(z,\xi)$ are non-constant odd functions such that $f_i(p_j) = 0$.. Furthermore, let ω be the form defined in 3.35 and let $\eta \in \bigoplus A^{(p,q)|(r,s)}$ be a form such that $(\overline{\partial} + i_V)(\eta) = 0$. If the number n (dimension) is even (odd), if the functions $\eta^{(0,0)|(n,n)}$, $\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ have only even (odd) quantities of variables ξ_j in its expansion and if $\det(\mathbb{B}(D))(\mathbb{B}(p_j)) = 1$, then:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\dots n;1\dots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n;1\dots n)}}{\mathrm{Ber}(V)}\right](p_j)\,.$$

Theorem 0.5. Let S be a compact complex supermanifold of $n|n$ dimension and let V be an odd holomorphic vector field with a non-degenerate isolated singularity $p_\kappa \in Sing(V)$, whose representation in local coordinates is equal to $V = \sum_{i=1}^n f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i \frac{\partial}{\partial \xi_i}$, where $g_i(z)$ are even functions without odd variables and $f_i(z,\xi)$ are odd functions such that $f_i(z,\xi) = \sum_{\lambda \in M} \xi^\lambda \cdot a^i_\lambda \cdot g_i + \sum_{\lambda\in M} \xi^\lambda \cdot b^i_\lambda \cdot g_{i_\lambda}$ with M being the set of multi-indices, $a^i_\lambda, b^i_\lambda \in \mathbb{C}$ and $i_\lambda \in \{1,\dots,\widehat{i},\dots,n\}$. Furthermore, let ω be the form defined in 3.35 and let $\eta \in \bigoplus A^{(p,q)|(r,s)}$ be a form such that $(\overline{\partial} + i_V)(\eta) = 0$. Then, if $\det(\mathbb{B}(D))(\mathbb{B}(p_\kappa)) = 1$, we have:

$$Res_{p_\kappa}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{\mathrm{Ber}(V)}\right](p_\kappa) + \left(\frac{2\pi}{i}\right)^n \left[\frac{\sum_{j=1}^n \sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)} \left\{a^j_\lambda \left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\dots n)} - \eta^{(0,0)|(n,n)}_{(\mu,1\dots n)}\right)\right\}}{\mathrm{Ber}(V)}\right](p_\kappa)\,.$$

where $L(\lambda)$ are odd numbers.

And as consequence of the previous results, we have the applications:

1. From the residue formula found at this thesis, *we deduced* the following Duistermaat-Heckman type formula for supermanifolds

$$\int_X e^{-sg}\frac{\omega^n}{n!} = \left(\frac{2\pi}{i}\right)^n \sum_{p_\kappa \in Sing(V)} \left[\frac{(e^{\omega - sg})^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)}\right](p_\kappa)\,;$$

2. From theorem 0.1 and from the formula (see [9], pg. 28)

$$\int_{\mathbb{P}^{n|m}} \omega_{FS}^n = \frac{1}{(n-m)!},$$

we conclude that a projective superspace $\mathbb{P}^{n|m}$, with $n \geq m$, has no vector field without singularities.

Notations

$\mathfrak{A}$	:	arbitrary abelian group;
$\mathcal{A}$	:	$\mathfrak{A}$-graded commutative ring with unit $1 \neq 0$
$\mathbb{C}_s$	:	complex Grassmann algebra;
$\mathbb{C}_s^{m,n}$	:	complex Grassmann superspace of dimension $m\vert n$;
S	:	complex supermanifold;
X	:	complex manifold associated to the supermanifold S;
$\mathbb{B}$	:	body map ;
$SB_\epsilon(p)$	:	superball centered on p of radius ϵ ;
$B_\epsilon(p)$	:	ball centered on p of radius ϵ;
$\omega, \eta, \ldots$	:	forms on Grassmann complex numbers ;
$\omega_\emptyset, \eta_\emptyset, \ldots$	:	forms on complex numbers associated to $\omega, \eta, \ldots$;
$A^{(p,q)\vert(r,s)}$	:	forms of type $f\ dz^1 \wedge \cdots \wedge dz^p \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q \wedge d\xi^1 \wedge \cdots \wedge d\xi^r \wedge d\bar{\xi}^1 \wedge \cdots \wedge d\bar{\xi}^s$.

Chapter 1

Graded commutative linear algebra

1.1 Graded commutative rings and graded modules

Notation 1.1. We will denote by $\mathfrak{A}$ an arbitrary abelian group.

Definition 1.2. An $\mathfrak{A}$-graded ring is a ring $\mathcal{A}$ with the additional property that there exists a family of subgroups $\mathcal{A}_\alpha$ (subgroups with respect to the additive (abelian) structure of the ring $\mathcal{A}$), $\alpha \in \mathfrak{A}$ such that:

- $\mathcal{A} = \bigoplus_{\alpha\in\mathfrak{A}} A_\alpha$
- $\forall\ \alpha, \beta \in \mathfrak{A} :\ \mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}$

The elements of $\mathcal{A}_\alpha$ are called *homogeneous elements of parity* α. For homogeneous elements, the parity map $\varepsilon : \bigcup_{\alpha\in\mathfrak{A}} \mathcal{A}_\alpha \to \mathfrak{A}$ is defined by $\varepsilon(\mathcal{A}_\alpha) = \alpha$.

Remark 1.3. Note that the parity map isn't defined on all ring $\mathcal{A}$.

Remark 1.4. Elements of zero parity will be called *even*.

Remark 1.5. The element $0 \in \mathcal{A}$ has ambiguous parity, i.e, $0 \in \mathcal{A}$ has any parity that we wish for because

$$0 \in \mathcal{A}_\alpha,\ \forall\ \alpha \in \mathfrak{A} \Rightarrow \varepsilon(0) = \alpha,\ \forall\ \alpha \in \mathfrak{A}.$$

Remark 1.6. Since $\mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta},\ \forall\, \alpha, \beta \in \mathfrak{A}$, then $\varepsilon(\mathcal{A}_\alpha . \mathcal{A}_\beta) = \varepsilon(\mathcal{A}_\alpha) + \varepsilon(\mathcal{A}_\beta) = \alpha + \beta$.

Lemma 1.7. If an $\mathfrak{A}$-graded ring $\mathcal{A}$ has unit $1 \neq 0$, then $\varepsilon(1) = 0$.

Proof. Vide [20, pg 3]. □

Definition 1.8. An $\mathfrak{A}$-graded commutative ring is an $\mathfrak{A}$-graded ring together with a symmetric bi-additive map $< \_|\_ >: \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{Z}_2 \simeq 0, 1$ such that:

$$a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta \Rightarrow\ a.b = (-1)^{<\alpha|\beta>} b.a$$

Notation 1.9. From now on we will consider $\mathcal{A}$ an $\mathfrak{A}$-graded commutative ring with unit $1 \neq 0$.

Definition 1.10. A $\mathfrak{A}$-graded left $\mathcal{A}$-module (respectively right) is a left $\mathcal{A}$-module E (respectively right) together with a family of subgroups E_α, $\alpha \in \mathfrak{A}$ (subgroups with respect to the additive (abelian) structure of E) satisfying:

(i) $E = \bigoplus_{\alpha \in \mathfrak{A}} E_\alpha$

(ii) $\mathcal{A}_\alpha . E_\beta \subset E_{\alpha+\beta}$ (respectively $E_\alpha . \mathcal{A}_\beta \subset E_{\alpha+\beta}$)

The elements of E_α are called homogeneous elements of parity α. The parity map

$$\varepsilon : \bigcup_{\alpha \in \mathfrak{A}} E_\alpha \to \mathfrak{A}$$

is defined by $\varepsilon(E_\alpha) = \alpha$.

Remark 1.11. The parity map is not defined for all E and $\varepsilon(0)$ has ambiguous parity $(0 \in E)$.

Remark 1.12. By $\mathcal{A}_\alpha . E_\beta \subset E_{\alpha+\beta}$, $\forall a \in \mathcal{A}_\alpha,\ \forall e \in E_\beta$ we have:

$$\varepsilon(a.e) = \varepsilon(a) + \varepsilon(e) = \alpha + \beta$$

Remark 1.13. Elements of zero parity will be called *even* elements.

Definition 1.14. A subset F of an $\mathfrak{A}$-graded left/right $\mathcal{A}$-module E is called an $\mathfrak{A}$-graded submodule if F is a submodule of the left/right $\mathcal{A}$-module E and if F, together with the subsets $F_\alpha = F \cap E_\alpha$, is itself a left/right $\mathfrak{A}$-graded $\mathcal{A}$-module.

Definition 1.15. By an $\mathfrak{A}$-graded $\mathcal{A}$-module we will always mean an $\mathfrak{A}$-graded $\mathcal{A}$-bimodule for which the left and right actions of $\mathcal{A}$ are related by:

$$e.a = (-1)^{\langle\alpha|\beta\rangle} a.e, \qquad \forall\ a \in \mathcal{A}_\alpha,\ \forall\ e \in E_\beta$$

Definition 1.16. A subset F of an $\mathfrak{A}$-graded $\mathcal{A}$-module E is called an $\mathfrak{A}$-graded submodule of E if it is an $\mathfrak{A}$-graded submodule of the left or right $\mathfrak{A}$-graded $\mathcal{A}$-module E.

The former definition is well defined because the following lemma.

Lemma 1.17. Given an $\mathfrak{A}$-graded $\mathcal{A}$-module E, let F be an $\mathfrak{A}$-graded submodule of the left structure (right structure) of E. Then F is too an $\mathfrak{A}$-graded submodule of the right structure (left structure) of E and it thus will be automatically an $\mathfrak{A}$-graded $\mathcal{A}$-module itself.

Proof. Vide [20, pg 6].

□

Definition 1.18. Let E be an $\mathfrak{A}$-graded left $\mathcal{A}$-module, let S be a subset of E and let B be a subset of $\mathcal{A}$. We define the subset $\operatorname{Span}_B(S) \subset E$ by:

$$\operatorname{Span}_B(S) = \left\{ \sum_{i=1}^{n} a^i s_i \mid n \in \mathbb{N},\ a^i \in B,\ s_i \in S \right\}$$

Remark 1.19. For an $\mathfrak{A}$-graded right $\mathcal{A}$-module E one just replace $\sum_{i=1}^{n} a^i s_i$ in the definition by $\sum_{i=1}^{n} s_i a^i$.

Remark 1.20. When $B = \mathcal{A}$ we just write $\operatorname{Span}(S)$.

Remark 1.21. It's easy to verify that $\operatorname{Span}(S)$ is a submodule of E, usually called *the submodule generated by* S.

Definition 1.22. Suppose that F_i, $i \in I$, be a family of submodules of E. Then:

$$\operatorname{Span}(\bigcup_{i\in I} F_i) = \left\{ \sum_{k=1}^{n} f_{i_k} \mid n \in \mathbb{N},\ i_k \in I,\ f_{i_k} \in F_{i_k} \right\}$$

is called *the sum of submodules* F_i.

Remark 1.23. $\operatorname{Span}(\bigcup_{i\in I} F_i)$ is usually denoted by $\sum_{i\in I} F_i$.

Lemma 1.24. If E is an $\mathfrak{A}$-graded $\mathcal{A}$-module and if F_i, $i \in I$, is a family of $\mathfrak{A}$-graded submodules of E, then $\sum_{i\in I} F_i$ is an $\mathfrak{A}$-graded submodule of E.

Proof. $f \in \sum_{i\in I} F_i \Rightarrow f = \sum_{k=1}^{n} f_{i_k}$, with $f_{i_k} \in F_{i_k}$. Since $\forall F_{i_k}$ is an $\mathfrak{A}$-graded submodule, then $f_{i_k} = \sum_{\alpha\in\mathfrak{A}} (f_{i_k})_\alpha$, with $(f_{i_k})_\alpha \in F_{i_k}$. Therefore, regrouping the homogeneous terms, we have:

$$f = \bigoplus_{\alpha\in\mathfrak{A}} f_\alpha$$

with $f_\alpha \in \operatorname{Span}(\bigcup_{i\in I} F_i),\ \forall \alpha \in \mathfrak{A}$. □

Lemma 1.25. If E is an $\mathfrak{A}$-graded $\mathcal{A}$-module and if $S \subset E$ is a set of homogeneous elements only, then $\mathrm{Span}(S)$ is an $\mathfrak{A}$-graded submodule of E.

Proof.

$$e \in \mathrm{Span}(S) \Rightarrow e = \sum_{i=1}^{n} a^i s_i = \sum_{i=1}^{n} \left(\sum_{\alpha\in\mathfrak{A}} a^i_\alpha \right) s_i, \quad \text{with } \varepsilon(s_i) \in \mathfrak{A}.$$

Now, regrouping the terms in order of parity, we find

$$e = \sum_{\beta\in\mathfrak{A}} \sum_{i=1}^{n} \sum_{\alpha\in\mathfrak{A}} \left(a^i_\alpha s_i \right)_\beta$$

that is exactly the decomposition of e in homogeneous elements. □

1.2 Multi-linear maps

Definition 1.26. Given the $\mathfrak{A}$-graded $\mathcal{A}$-modules $E_1, \dots, E_k$ and F, a k-additive map $\phi : E_1 \times \cdots \times E_k \to F$ is said to be left k-linear if $\forall i$, $\forall e_i \in E_i$ and $\forall a \in \mathcal{A}$, we have:

(i) $\phi(e_1, \dots, e_{i-1}, e_i a, e_{i+1}, \dots, e_k) = \phi(e_1, \dots, e_{i-1}, e_i, a e_{i+1}, \dots, e_k)$

(ii) $\phi(a e_1, e_2, \dots, , e_k) = a\phi(e_1, e_2, \dots, , e_k)$

The ϕ map is called right k-linear if the condition (ii) is modified to:

(ii)' $\phi(e_1, \dots, e_{k-1}, e_k a) = \phi(e_1, \dots, e_{k-1}, e_k)a$

Definition 1.27.

(•) The set of all left k-linear maps is denoted by $\mathrm{Map}_L\left(E_1, \dots, E_k; F\right)$

(•) The set of all right k-linear maps is denoted by $\mathrm{Map}_R\left(E_1, \dots, E_k; F\right)$

(•) $\mathrm{Map}_S\left(E_1, \dots, E_k; F\right) = \mathrm{Map}_L\left(E_1, \dots, E_k; F\right)$ or $\mathrm{Map}_R\left(E_1, \dots, E_k; F\right)$

Definition 1.28. The map $\phi \in \mathrm{Map}_S\left(E_1, \dots, E_k; F\right)$ is called a k-linear map of parity $\alpha \in \mathfrak{A}$ if:

$$\phi((E_1)_{\beta_1}, \dots, (E_k)_{\beta_k}) \subset F_{\alpha+\beta_1+\cdots+\beta_k}, \quad \forall \beta_i \in \mathfrak{A}$$

Definition 1.29. $\mathrm{Map}_S\left(E_1, \dots, E_k; F\right)_\alpha \subset \mathrm{Map}_S\left(E_1, \dots, E_k; F\right)$ denote the subset of all k-linear maps of parity α.

Definition 1.30.

(•) For left k-linear maps, the right multiplication is defined by:

$$(\phi a)(e_1, \dots, e_k) = [\phi(e_1, \dots, e_k)]a$$

(•) For right k-linear maps, the left multiplication is defined by:

$$(a\phi)(e_1, \dots, e_k) = a[\phi(e_1, \dots, e_k)]$$

Lemma 1.31. With the former definitions, the set $\mathrm{Map}_L\,(E_1, \dots, E_k; F)$ becomes a right $\mathcal{A}$-module and the set $\mathrm{Map}_R\,(E_1, \dots, E_k; F)$ becomes a left $\mathcal{A}$-module.

Proof. Vide [20, pg 8]. □

Lemma 1.32. $\forall\ \alpha, \beta \in \mathfrak{A}$ and $\phi \in \mathrm{Map}_S\,(E_1, \dots, E_k; F)_\alpha$, $\psi \in \mathrm{Map}_S\,(F; H)_\beta$, then we have

$$\psi \circ \phi \in \mathrm{Map}_S\,(E_1, \dots, E_k; H)_{\alpha+\beta}$$

Proof. For $e_1, \dots, e_k$ homogeneous, we have:

$$\varepsilon(\psi \circ \phi(e_1, \dots, e_k)) = \varepsilon(\psi) + \varepsilon(\phi(e_1, \dots, e_k)) = \beta + \varepsilon(\phi) + \sum_{i=1}^{k} \varepsilon(e_i) = \beta + \alpha + \sum_{i=1}^{k} \varepsilon(e_i).$$

□

Definition 1.33. A k-linear map $\phi \in \mathrm{Map}_S\,(E_1, \dots, E_k; F)$ is called a (homo)morphism if this map is a finite sum of homogeneous k-linear maps. More precisely, we define the set $\mathrm{Hom}_S\,(E_1, \dots, E_k; F)$ of all k-linear homomorphisms by:

$$\mathrm{Hom}_S\,(E_1, \dots, E_k; F) = \sum_{\alpha \in \mathfrak{A}} \mathrm{Map}_S\,(E_1, \dots, E_k; F)_\alpha \subset \mathrm{Map}_S\,(E_1, \dots, E_k; F)\,.$$

Remark 1.34. When all $\mathfrak{A}$-graded $\mathcal{A}$-modules coincide, we denote $\mathrm{Hom}_S\,(E_1, \dots, E_k; F)$ by $\mathrm{Hom}_S\left(E^k; F\right)$.

Remark 1.35. $\mathrm{Hom}_S\,(E; E)$ is denoted as $\mathrm{End}_S(E)$ and its elements are called *endomorphisms of E.*

Remark 1.36.

(•) $\mathrm{Hom}_L\,(E; \mathcal{A})$ is denoted by ${}^*E$ and is called the *left dual of E.*

(•) $\mathrm{Hom}_R\,(E; \mathcal{A})$ is denoted by E^* and is called the *right dual of E.*

Lemma 1.37. The left/right $\mathcal{A}$-module $\mathrm{Hom}_S(E_1, \ldots, E_k; F)$ together with its subsets $\mathrm{Hom}_S(E_1, \ldots, E_k; F)_\alpha$ is an $\mathfrak{A}$-graded $\mathcal{A}$-module. If the abelian group $\mathfrak{A}$ is finite, then $\mathrm{Hom}_S(E_1, \ldots, E_k; F) = \mathrm{Map}_S(E_1, \ldots, E_k; F)$.

Proof. Vide [20, pg 9].

□

Corollary 1.38. $\mathit{Hom}_L(E_1, \ldots, E_k; F)_0 = \mathit{Hom}_R(E_1, \ldots, E_k; F)_0$

Proof. Vide [20, pg 10]. □

Remark 1.39. As $\mathrm{Hom}_L(E_1, \ldots, E_k; F)_0 = \mathrm{Hom}_R(E_1, \ldots, E_k; F)_0$, then we can just write $\mathrm{Hom}(E_1, \ldots, E_k; F)_0$ for both.

Definition 1.40.

(•) An even invertible linear map $\phi \in \mathrm{Hom}(E; F)_0$ is called an *isomorphism between E and F*. If there exists an isomorphism between E and F, then they are called *isomorphic*.

(•) An even invertible endomorphism $\psi \in \mathrm{End}(E)_0$ is called an *automorphism of E*. The set of all automorphisms of E is denoted by $\mathrm{Aut}(E)$.

1.3 Free $\mathfrak{A}$-graded $\mathcal{A}$-modules and quotients

Definition 1.41 (free $\mathfrak{A}$-graded $\mathcal{A}$-module). Let $\varepsilon : G \to \mathfrak{A}$ be a map from an abstract set G to $\mathfrak{A}$, and define $G_\alpha \subset G$ by $G_\alpha = \varepsilon^{-1}(\alpha)$. We define the space $F(G, \varepsilon)$ as the set of all maps $f : G \to \mathcal{A}$ with the property that $f(g) = 0$ for all $g \in G$ except finitely many. In $F(G, \varepsilon)$, we define the addition by (f+f')(g)=f(g)+f'(g), and a (left) multiplication by elements of $\mathcal{A}$ by $(af)(g) = af(g)$. In this way $F(G, \varepsilon)$ becomes a left $\mathcal{A}$-module. We identify each element $g \in G$ with the map $\phi_g : G \to \mathcal{A}$ defined by $\phi_g(g) = 1$ and $\phi_g(h) = 0$ for $h \neq g$. Thus, each $f \in F(G, \varepsilon)$ can be written in a unique way as:

$$f = \sum_{g \in G} f^g \phi_g \equiv \sum_{g \in G} f^g \cdot g$$

where $f^g \in \mathcal{A}$ is defined by $f^g = f(g)$.

Remark 1.42. By definition, the sum above need to be finite.

Definition 1.43.

$$F(G, \varepsilon)_\alpha := \left\{ f \in F(G, \varepsilon) \mid \forall g \in G : \; f(g) \in \mathcal{A}_{\alpha - \varepsilon(g)} \right\}$$

Lemma 1.44. $f = \sum_{g \in G} f^g \cdot g$ has parity α if and only if f^g has parity $\alpha - \varepsilon(g)$, $\forall g \in G$. Furthermore, for $\phi_g \in F(G, \varepsilon)$ we have $\varepsilon(\phi_g) = \varepsilon(g)$.

Proof.

(i) $f^g = f(g) \in \mathcal{A}_{\alpha - \varepsilon(g)}, \ \forall g \in G \Leftrightarrow \varepsilon(f^g) = \alpha - \varepsilon(g), \ \forall g \in G$.

(ii) $\phi_g(g) = 1 \in \mathcal{A}_0 = \mathcal{A}_{\alpha - \varepsilon(g)}$ and, $\forall h \neq g$, $\phi_g(h) = 0 \in \mathcal{A}_0 = \mathcal{A}_{\alpha - \varepsilon(g)} \Rightarrow \varepsilon(\phi_g) = \varepsilon(g)$. □

Decomposing each $f^g \in \mathcal{A}$ in its unique homogeneous components, we get (regrouping the elements) to decompose any $f \in F(G, \varepsilon)$ in unique homogeneous parts. Therefore:

$$F(G, \varepsilon) = \bigoplus_{\alpha \in \mathfrak{A}} F(G, \varepsilon)_\alpha$$

Furthermore, we have $\mathcal{A}_\alpha \cdot F(G, \varepsilon)_\beta \subset F(G, \varepsilon)_{\alpha+\beta}$. So $F(G, \varepsilon)$ is an $\mathfrak{A}$-graded $\mathcal{A}$-module.

Remark 1.45. $F(G, \varepsilon)$ is usually called the *free $\mathfrak{A}$-graded $\mathcal{A}$-module.*

Definition 1.46 (quotients). Let E be an $\mathfrak{A}$-graded $\mathcal{A}$-module and let F be an $\mathfrak{A}$-graded submodule of E. The quotient $G = E/F$ with canonical projection $\pi : E \to G$ is defined by:

$$\pi(e) = \pi(e') \Leftrightarrow e - e' \in F.$$

The addition and (left) multiplication by elements in $\mathcal{A}$ is defined by:

(•) $\pi(e) + \pi(e') = \pi(e + e')$

(•) $a\pi(e) = \pi(ae)$

The subgroups G_α are defined by:

$$\pi(e) \in G_\alpha \Leftrightarrow \exists f \in F : \ e - f \in E_\alpha$$

Remark 1.47. By the above definition, we deduce that π is an even linear map, i.e., $\pi \in \mathrm{Hom}_S\,(E; G)_0$.

Lemma 1.48. Let E be an $\mathfrak{A}$-graded $\mathcal{A}$-module and let F be an $\mathfrak{A}$-graded submodule of E. Then, by above definitions, we have that $G = E/F$ is an $\mathfrak{A}$-graded $\mathcal{A}$-module.

Proof. It is clear that G is a $\mathcal{A}$-module and that $\mathcal{A}_\alpha G_\beta = G_{\alpha+\beta}$. Now, for all $e \in E$, we have:

$$\pi(e) = \pi(\sum_{\alpha \in \mathfrak{A}} e_\alpha) = \sum_{\alpha \in \mathfrak{A}} \pi(e_\alpha)$$

with $\pi(e_\alpha) \in G_\alpha$. It remains to prove that this decomposition is unique. For this, it is enough to show that $\sum_{\alpha\in\mathfrak{A}} \pi(e_\alpha) = \bar{0} \Rightarrow \pi(e_\alpha) = \bar{0},\ \forall\alpha \in \mathfrak{A}$. Indeed: $\sum_{\alpha\in\mathfrak{A}} \pi(e_\alpha) = \pi(\sum_{\alpha\in\mathfrak{A}} e_\alpha) = \bar{0} \Rightarrow \sum_{\alpha\in\mathfrak{A}} e_\alpha = f \in F$. Since F is an $\mathfrak{A}$-graded submodule of E, we have that $e_\alpha \in F,\ \forall\alpha \in \mathfrak{A} \Rightarrow \pi(e_\alpha) = \bar{0},\ \forall\alpha \in \mathfrak{A}$. □

1.4 Tensor products

Let E and F be $\mathfrak{A}$-graded $\mathcal{A}$-modules and consider the set $G = (\cup_{\alpha\in\mathfrak{A}} E_\alpha \backslash \{0\}) \times (\cup_\beta F_\beta \backslash \{0\}) \subset E \times F$,i.e., G is the product of all non-zero homogeneous elements in E and in F. On G we define a parity map $\varepsilon : G \to \mathfrak{A}$ by $\varepsilon(e, f) = \varepsilon(e) + \varepsilon(f)$. Let $F(G, \varepsilon)$ be the free $\mathfrak{A}$-graded $\mathcal{A}$-module associated to G. We define the subset S of $F(G, \varepsilon)$ as the union of two subsets: $S = S_a \cup S_m$ with:

$$S_a = \{\phi_{(e+e',f)} - \phi_{(e,f)} - \phi_{(e',f)},\ \phi_{(e,f+f')} - \phi_{(e,f)} - \phi_{(e,f')} | \forall\ \alpha, \beta \in \mathfrak{A},\ e, e' \in E_\alpha,\ f, f' \in F_\beta\} \tag{1.1}$$

$$S_m = \{\phi_{(ae,f)} - a\phi_{(e,f)},\ \phi_{(ea,f)} - \phi_{(e,af)} | \forall\ \alpha, \beta, \gamma \in \mathfrak{A},\ a \in \mathcal{A}_\alpha,\ e \in E_\beta,\ f \in F_\gamma\}. \tag{1.2}$$

Definition 1.49.

$$E \otimes F := F(G, \varepsilon)/\mathrm{Span}(S).$$

This $\mathfrak{A}$-graded $\mathcal{A}$-module is called *the tensor product of E and F.*

Remark 1.50. $E \otimes F$ is indeed an $\mathfrak{A}$-graded $\mathcal{A}$-module because of lemmas 1.25 and 1.48.

Definition 1.51. We define the map $\chi : E \times F \to E \otimes F$ as:

$$\chi(e, f) = \sum_{\alpha,\beta\in\mathfrak{A}} \pi\left(\phi_{(e_\alpha, f_\beta)}\right). \tag{1.3}$$

Lemma 1.52. The map $\chi : E \times F \to E \otimes F$ is even and bilinear.

Proof. Vide [20, pg 18]. □

Remark 1.53. By induction, we find $\chi(e_1, \ldots, e_k) = e_1 \otimes \cdots \otimes e_k$.

Proposition 1.54. Given any $\mathfrak{A}$-graded $\mathcal{A}$-module H and any $\psi \in \mathrm{Map}_S(E, F; H)$, there exists a unique $\Psi \in \mathrm{Map}_S(E \otimes F; H)$ such that $\psi = \Psi \circ \chi$. If ψ has parity α, then so has Ψ.

Proof. Vide [20, pg 19]. □

Remark 1.55. The above result can be generalized to any k-linear maps, i.e., given $\psi \in \mathrm{Map}_S(E_1, \ldots, E_k; H)$, then there exists a unique linear map $\Psi \in \mathrm{Map}_S(E_1 \otimes \cdots \otimes E_k; H)$ such that $\psi = \Psi \circ \chi$ and if $\varepsilon(\psi) = \alpha$, then $\varepsilon(\Psi) = \alpha$.

Notation 1.56. For $E_1, \ldots, E_k$ $\mathfrak{A}$-graded $\mathcal{A}$-modules, we have:

$$\bigotimes_{i=1}^{k} E_i = E_1 \otimes \cdots \otimes E_k$$

$$\bigotimes_{i=k}^{1} E_i = E_k \otimes \cdots \otimes E_1$$

$$\bigotimes^{k} E = E \otimes \cdots \otimes E$$

$$\bigotimes^{0} E = \mathcal{A} \qquad \text{(formal definition)}$$

$$\left(\bigotimes^{K} E\right) \otimes \left(\bigotimes^{L} E\right) = \bigotimes^{K+L} E \qquad \text{with } K, L \in \mathbb{N}$$

For $K = 0$ or $L = 0$, we will use the isomorphism $\underline{m}_L$ or $\underline{m}_R$, i.e

$$\left(\bigotimes^{0} E\right) \otimes \left(\bigotimes^{L} E\right) = \mathcal{A} \otimes \left(\bigotimes^{L} E\right) \cong \bigotimes^{L} E$$

$$\left(\bigotimes^{K} E\right) \otimes \left(\bigotimes^{0} E\right) = \left(\bigotimes^{K} E\right) \otimes \mathcal{A} \cong \bigotimes^{K} E$$

Definition 1.57. For any two $\mathfrak{A}$-graded $\mathcal{A}$-module E, F, we define $R : E \times F \to F \otimes E$ by:

$$R(e, f) = \sum_{\alpha,\beta \in \mathfrak{A}} (-1)^{\langle \alpha|\beta \rangle} \chi(f_\beta, e_\alpha)$$

or

$$R(e, f) = \sum_{\alpha,\beta \in \mathfrak{A}} (-1)^{\langle \alpha|\beta \rangle} f_\beta \otimes e_\alpha$$

Lemma 1.58. R is an even bilinear map.

Proof.

(•) R is even. Indeed, for e, f homogeneous, we have:

$$\varepsilon(R(e, f)) = \varepsilon\left((-1)^{\langle \alpha|\beta \rangle} f_\beta \otimes e_\alpha\right) = \varepsilon(e_\alpha) + \varepsilon(f_\beta) = \alpha + \beta$$

(•) Let's prove the additivity in the first entrance. In the second, it's similar.
$R(e_1+e_2,f)=\sum_{\alpha,\beta\in\mathfrak{A}}(-1)^{\langle\alpha|\beta\rangle}\chi(f_\beta,e_{1_\alpha}+e_{2_\alpha})=$
$\sum_{\alpha,\beta\in\mathfrak{A}}(-1)^{\langle\alpha|\beta\rangle}\chi(f_\beta,e_{1_\alpha})+\sum_{\alpha,\beta\in\mathfrak{A}}(-1)^{\langle\alpha|\beta\rangle}\chi(f_\beta,e_{2_\alpha})=R(e_1,f)+R(e_2,f)$.

(•) $R(ea,f)=R(e,af)$. Indeed, by additivity, choose e,f,a homogeneous, then:
$R(ea,f)=(-1)^{\langle\varepsilon(e)+\varepsilon(a)|\varepsilon(f)\rangle}f\otimes ea=(-1)^{\langle\varepsilon(e)|\varepsilon(a)+\varepsilon(f)\rangle}af\otimes e=R(e,af)$.

(•) $R(ae,f)=aR(e,f)$. Indeed:
$R(ae,f)=(-1)^{\langle\varepsilon(e)+\varepsilon(a)|\varepsilon(f)\rangle}f\otimes ae=(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}af\otimes e=a\Big((-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}f\otimes e\Big)=$
$a\mathbb{R}(e,f)$.

□

Definition 1.59. The even linear map $\mathfrak{R}:E\otimes F\to F\otimes E$, induced from R, is given by:

$$\mathfrak{R}(e\otimes f)=(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}f\otimes e\qquad\forall\ e,f\text{ homogeneous.}$$

This map is called *the interchanging map of E and F*.

Lemma 1.60. $\mathfrak{R}$ is an isomorphism between $E\otimes F$ and $F\otimes E$.

Proof. $\forall\ e,f$ homogeneous, define $\mathfrak{R}^{-1}:F\otimes E\to E\otimes F$ by:

$$\mathfrak{R}^{-1}(f\otimes e)=(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}e\otimes f.$$

This map is linear, even and satisfies the following:

$$\mathfrak{R}^{-1}\circ\mathfrak{R}(e\otimes f)=(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}\mathfrak{R}^{-1}(f\otimes e)=(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}(-1)^{\langle\varepsilon(e)|\varepsilon(f)\rangle}e\otimes f=e\otimes f.$$

Analogously, $\mathfrak{R}\circ\mathfrak{R}^{-1}(f\otimes e)=f\otimes e$. □

Remark 1.61. From lemma above, one concludes that $E\otimes F\cong F\otimes E$.

Definition 1.62. We define

$$\mathfrak{R}_{(ii+1)}:E_1\otimes\cdots E_i\otimes E_{i+1}\otimes\cdots\otimes E_k\to E_1\otimes\cdots\otimes E_{i+1}\otimes E_i\otimes\cdots\otimes E_k$$

by $\mathfrak{R}_{(ii+1)}(e_1\otimes\cdots\otimes e_i\otimes e_{i+1}\otimes\cdots\otimes e_k)=e_1\otimes\cdots\otimes\mathfrak{R}(e_i\otimes e_{i+1})\otimes\cdots\otimes e_k=$
$(-1)^{\langle\varepsilon(e_i)|\varepsilon(e_{i+1})\rangle}e_1\otimes\cdots\otimes e_{i+1}\otimes e_i\otimes\cdots\otimes e_k$.

1.5 Exterior powers

Definition 1.63. A k-linear map $\phi:E^k\to F$, where E and F are $\mathfrak{A}$-graded $\mathcal{A}$-modules, is called $\mathfrak{A}$-*graded skew-symmetric* if for all homogeneous $e_i\in E$ and all $j=1,\ldots,k-1$

we have:

$$\phi(e_1,\dots,e_j,e_{j+1},\dots,e_k) = -(-1)^{\langle\varepsilon(e_j)|\varepsilon(e_{j+1})\rangle}\phi(e_1,\dots,e_{j+1},e_j,\dots,e_k)$$

Notation 1.64.

(•) $\mathrm{Map}_S^{sk}\left(E^k;F\right)$: set of all (left or right) $\mathfrak{A}$-graded skew-symmetric k-linear maps.

(•) $\mathrm{Hom}_S^{sk}\left(E^k;F\right)$: $\mathrm{Map}_S^{sk}\left(E^k;F\right)\bigcap \mathrm{Hom}_S\left(E^k;F\right)$.

Remark 1.65. A k-linear map $\phi : E^k \to F$ is said to be $\mathfrak{A}$-graded skew-symmetric if, and only if, the induced linear map $\Phi : \bigotimes^k E \to F$ satisfies, for $j = 1,\dots,k-1$, the following relation:

$$\Phi = -\Phi\circ\mathfrak{R}_{jj+1} \quad \text{or} \quad \Phi\left(Id + \mathfrak{R}_{jj+1}\right) = 0.$$

Remark 1.66. If $\Phi = \Phi\circ\mathfrak{R}_{jj+1}$, for all $j = 1,\dots,k-1$, then we said that ϕ is $\mathfrak{A}$-*graded symmetric*.

Definition 1.67. Let E be an $\mathfrak{A}$-graded $\mathcal{A}$-module and let $T_k \subset \bigotimes^k E$ be a set defined by:

$$T_k = \left\{e_1\otimes\cdots\otimes e_k + \mathfrak{R}_{(jj+1)}(e_1\otimes\cdots\otimes e_k) | 1 \le j < k,\ e_i \in E \text{ homogeneous } \forall i\right\}.$$

Since T_k is composed by homogeneous elements, then $\mathrm{Span}(T_k)$ is an $\mathfrak{A}$-graded submodule of $\bigotimes^k E$ (lemma 1.25).

Definition 1.68.

$$\bigwedge^k E = \bigotimes^k E/\mathrm{Span}(T_k)$$

$\bigwedge^k E$ is called *the k-th exterior power of the $\mathfrak{A}$-graded $\mathcal{A}$-module E.*

Remark 1.69. By lemma 1.48, we conclude that $\bigwedge^k E$ is an $\mathfrak{A}$-graded $\mathcal{A}$-module.

Definition 1.70. Given the canonical projection $\pi : \bigotimes^k E \to \bigotimes^k E/\mathrm{Span}(T_k) \equiv \bigwedge^k E$, we define the even k-linear map $\omega : E^k \to \bigwedge^k E$ by:

$$\omega = \pi\circ\chi.$$

Definition 1.71.

(•) We define $\mathrm{Span}(T_1) = \{0\}$. As $\bigotimes^1 E = E$, we have $\bigwedge^1 E = E$ and $\omega : E^1 \to \bigwedge^1 E\,(= E)$. Therefore $\omega = Id(E)$.

(•) We define $\mathrm{Span}(T_0) = \{0\}$. As $\bigotimes^0 E = \mathcal{A}$, we have $\bigwedge^0 E = \mathcal{A}$ and $\omega : E^0 \,(= \mathcal{A}) \to \bigwedge^0 E \,(= \mathcal{A})$. Therefore $\omega = Id(\mathcal{A})$.

Proposition 1.72. Let E be an $\mathfrak{A}$-graded $\mathcal{A}$-module and $k \geq 1$, then $\omega : E^k \to \wedge^k E$ is $\mathfrak{A}$-graded skew-symmetric. Moreover, given any $\mathfrak{A}$-graded $\mathcal{A}$-module F and any map $\phi \in \mathrm{Map}_S^{sk}\left(E^k; F\right)$, there exists a unique map $\Phi \in \mathrm{Map}_S\left(\wedge^k E; F\right)$ such that $\phi = \Phi \circ \omega$. If ϕ has parity α, then so has Φ.

Proof. Vide [20, pg 25]. □

Definition 1.73. There exists a unique even linear map $\wedge : \bigwedge^k E \times \bigwedge^l E \to \bigwedge^{k+l} E$ called *the wedge product*, such that the following equivalence is true:

$$\left(\bigwedge^k E\right) \wedge \left(\bigwedge^l E\right) = \omega\left(\bigwedge^k E, \bigwedge^l E\right) \cong \bigotimes^{k+l} E/\mathrm{Span}(T_{k+l})$$

Remark 1.74. Since $\mathcal{A} = \wedge^0 E$, then for $a \in \wedge^0 E$, we will have:

$$a \wedge e_1 \wedge \cdots \wedge e_k = ae_1 \wedge \cdots \wedge e_k \text{ and } e_1 \wedge \cdots \wedge e_k \wedge a = e_1 \wedge \cdots \wedge e_k a$$

Proposition 1.75. Given an $\mathfrak{A}$-graded $\mathcal{A}$-module E, $k, l \in \mathbb{N}$ and $K \in \bigwedge^k E$, $L \in \bigwedge^l E$, both homogeneous, then:

$$K \wedge L = (-1)^{kl} \cdot (-1)^{\langle \varepsilon(K)|\varepsilon(L)\rangle} L \wedge K.$$

Proof. Vide [20, pg 27]. □

Definition 1.76. The exterior algebra $\bigwedge E$ of an $\mathfrak{A}$-graded $\mathcal{A}$-module E is defined as the direct sum:

$$\bigwedge E = \bigoplus_{k=0}^{\infty} \wedge^k E.$$

1.6 $\mathbb{Z}_2$-graded algebra

Remark 1.77. Throughout this chapter, we will use (as will be seen) $\mathcal{A}$ as an $\mathbb{R}$-algebra and $\mathcal{A}/\mathcal{N} \cong \mathbb{R}$, but all the results are also valid for $\mathcal{A}$ a $\mathbb{C}$-algebra and $\mathcal{A}/\mathcal{N} \cong \mathbb{C}$.

Definition 1.78. From now on we will consider $\mathfrak{A} = \mathbb{Z}_2$. The elements of parity zero will be called *even elements* and elements of parity one *odd elements*. The symmetric bilinear map $\langle \_|\_ \rangle : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ will be given by

$$\langle \alpha|\beta \rangle = \alpha \cdot \beta$$

Definition 1.79. For any $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}$ we define the *set of nilpotent elements* $\mathcal{N}$ by:

$$\mathcal{N} = \{a \in \mathcal{A} \mid \exists k \in \mathbb{N} : a^k = 0\}.$$

Example 1.80. Let X be a real vectorial space (of finite or infinite dimension) and let $\mathcal{A}$ be the exterior algebra $\mathcal{A} = \bigwedge X$. $\mathcal{A}$ become a graded commutative ring if we define:

$$\mathcal{A}_0 = \bigoplus_{k=0}^{\infty} \bigwedge^{2k} X; \ \mathcal{A}_1 = \bigoplus_{k=0}^{\infty} \bigwedge^{2k+1} X.$$

In this case, we have $1 \in \bigwedge^0 X \cong \mathbb{R}$ and $\mathcal{N} = \bigoplus_{k=1}^{\infty} \bigwedge^k X$.

Lemma 1.81. Let $\mathcal{A}$ be an graded commutative ring and let $\mathcal{N} \subset \mathcal{A}$ be the set of nilpotent elements. Then $\mathcal{N} = (\mathcal{N} \cap \mathcal{A}_0) \oplus \mathcal{A}_1$ and $\mathcal{N}$ is an ideal of $\mathcal{A}$.

Proof. Vide [20, pg 56]. □

Lemma 1.82. If $n_1, \ldots, n_N$ is a finite number of nilpotent elements in $\mathcal{A}$, then there exists a non-zero homogeneous nilpotent n such that $\forall i : nn_i = 0$.

Proof. Vide [20, pg 57]. □

Definition 1.83. If $\mathcal{A}$ is a graded commutative ring, we denote by $\mathbb{B}$ the canonical projection $\mathbb{B} : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ and call it the *body map*. Since $\mathcal{N}$ is an ideal, we have that $\mathcal{A}/\mathcal{N}$ is a ring, and $\mathbb{B}$ will be a ring homomorphism, i.e.:

- $\mathbb{B}(a_1 \cdot a_2) = \mathbb{B}(a_1) \cdot \mathbb{B}(a_2)$;
- $\mathbb{B}(a_1 + a_2) = \mathbb{B}(a_1) + \mathbb{B}(a_2)$.

Definition 1.84. If the graded commutative ring $\mathcal{A}$ is a $\mathbb{R}$-algebra ($\mathbb{C}$-algebra), then we have in particular that $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{N}$, $\mathcal{A}/\mathcal{N}$ are vector spaces over $\mathbb{R}$ (over $\mathbb{C}$) and $\mathbb{B} : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ will be a linear map between vector spaces over $\mathbb{R}$ (over $\mathbb{C}$).

Definition 1.85. From now on $\mathcal{A}$ will be a graded commutative $\mathbb{R}$-algebra ($\mathbb{C}$-algebra) with unit, such that $\mathcal{A}/\mathcal{N} \cong \mathbb{R}$ ($\mathcal{A}/\mathcal{N} \cong \mathbb{C}$).

Remark 1.86. For $a \in \mathcal{A}$, we write $a = r + n$, with $\mathbb{B}(a) = r$ and $n \in \mathcal{N}$.

Example 1.87. The complex Grassmann algebra $\mathbb{C}_{S[L]}$ is a graded commutative $\mathbb{C}$-algebra such that $\mathbb{C}_{S[L]}/\mathcal{N} \cong \mathbb{C}$.

Definitions 1.88.

(•) The set $M(m \times n, \mathcal{A})$ denotes the set of all matrices of size $m \times n$ with entries in $\mathcal{A}$;

(•) The usual matrix multiplication $M(m \times n, \mathcal{A}) \times M(n \times r, \mathcal{A}) \to M(m \times r, \mathcal{A})$ still makes sense on these sets;

(•) The body map $\mathbb{B}$ extends in a natural way to these matrices: $\mathbb{B} : M(m \times n, \mathcal{A}) \to M(m \times n, \mathbb{R})$ or $\mathbb{B} : M(m \times n, \mathcal{A}) \to M(m \times n, \mathbb{C})$, and this map is surjective and preserves matrix multiplication;

(•) The set $M(m \times n, \mathcal{A})$ equipped with matrix multiplication is a ring with unit In;

(•) $\mathbb{B} : M(m \times n, \mathcal{A}) \to M(m \times n, \mathbb{R} \text{ or } \mathbb{C})$ is a surjective ring homomorphism.

(•) For $a \in M(m \times n, \mathcal{A})$ we define the *rank of* a, denoted as $\operatorname{rank}(a)$, as the rank of its body $\mathbb{B}(a) \in M(m \times n, \mathbb{R} \text{ or } \mathbb{C})$, i.e, $\operatorname{rank}(a) = \operatorname{rank}(\mathbb{B}(a))$. In other words, $\operatorname{rank}(a)$ is the number of independent rows or columns in $\mathbb{B}(a)$.

Lemma 1.89. An element $a \in M(n \times n, \mathcal{A})$ is invertible if and only if $\operatorname{Det}(\mathbb{B}(a)) \neq 0$.

Proof. Vide [20, pg 58].

□

Definition 1.90. We define $Gl(p|q, \mathcal{A}) \subset M((p+q) \times (p+q))$ as being an especial set of invertible matrices such that if $X \in Gl(p|q, \mathcal{A})$ then $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a invertible matrix whose blocks A and D are composed exclusively by even elements and the blocks B and C are composed exclusively by odd elements.

Remark 1.91. The block A above has order equal to $p \times p$.

1.7 Free graded $\mathcal{A}$-modules

Definition 1.92. We will call a free graded $\mathcal{A}$-module *finite dimensional* if it admits a finite (homogeneous) basis. A homogeneous basis $e_1, \ldots, e_k$ of a finite dimensional free graded $\mathcal{A}$-module is called *ordered* if all even vectors come first, i.e., e_i even and e_j odd implies $i < j$.

Definition 1.93. A subset F of a free graded $\mathcal{A}$-module E is called a *graded subspace* if it is a graded submodule of E that in itself is a free graded $\mathcal{A}$-module.

Proposition 1.94. Let $e_1 \ldots, e_n$ be a basis of a graded $\mathcal{A}$-module E.

(i) If $f_1, \ldots, f_m$ is another basis of E, then m=n.

(ii) All other bases $\{f_1, \ldots, f_n\}$ are classified by invertible matrices $a \in M(n \times n, \mathcal{A})$ with $f_i = \sum_j a_i^j e_j$.

(iii) If $f_1, \ldots, f_n$ is either generating or independent, it is a basis.

Proof. Vide [20, pg 60]. □

Proposition 1.95. The number of even vectors in a homogeneous basis of a finite dimensional free graded $\mathcal{A}$-module E is an invariant of E.

Proof. Vide [20, pg 61]. . □

Definition 1.96. The *graded dimension* of a finite dimensional free graded $\mathcal{A}$-module E is a pair of integers (p, q) where p, called *the even dimension of* E, is the number of even vectors in a basis for E and q, called *the odd dimension of* E, is the number of odd vectors in bases. We usually denote this as $\dim(E) = p|q$.

1.8 The Berezinian

The ordinary determinant is defined on square matrices with reals or complex coefficients. We will see that is possible extend this concept to matrices $Gl(p|q, \mathcal{A})$ through the definition of a graded determinant $\mathrm{Ber} : Gl(p|q, \mathcal{A}) \to \mathcal{A}_0$ called *Berezinian determinant.*

Definition 1.97. Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Gl(p|q, \mathcal{A})$, then we define *the Berezinian determinant* $\mathrm{Ber} : Gl(p|q, \mathcal{A}) \to \mathcal{A}_0$ by:

$$\mathrm{Ber}(X) = \mathrm{Det}\left(A - BD^{-1}C\right)(\mathrm{Det}\,(D))^{-1}.$$

Remark 1.98. Observe that $\mathrm{Det}\,(D) \neq 0$ and the definition of the Berezinian is well placed. Indeed, since $\forall X \in Gl(p|q, \mathcal{A})$ is in particular invertible, we have $\mathrm{Det}\,(\mathbb{B}\,(X)) \neq 0$ $\implies \mathrm{Det}\,(\mathbb{B}\,(A)) \neq 0$ and $\mathrm{Det}\,(\mathbb{B}\,(D)) \neq 0$ (by the classical theory). Therefore, $\mathrm{Det}\,(D) \neq 0$.

Remark 1.99. If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Gl(p|q, \mathcal{A}) \implies$ the blocks A and D are invertible.

Proposition 1.100. The map $\mathrm{Ber} : Gl(p|q, \mathcal{A}) \to \mathcal{A}_0$ is a homomorphism, i.e:

$$\mathrm{Ber}\left(X\widehat{X}\right) = \mathrm{Ber}\,(X)\,\mathrm{Ber}\left(\widehat{X}\right).$$

Proof. Vide [20, pg 78]. □

Corollary 1.101.

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}\,(A - BD^{-1}C)}{\mathrm{Det}\,(D)} = \frac{\mathrm{Det}\,(A)}{\mathrm{Det}\,(D - CA^{-1}B)}$$

Proof. The first equality come from the definition. We will prove the second. Consider the following decomposition:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_p & A^{-1}B \\ D^{-1}C & I_q \end{pmatrix}$$

By proposition 1.100, we have:

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Ber}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \cdot \mathrm{Ber}\begin{pmatrix} I_p & A^{-1}B \\ D^{-1}C & I_q \end{pmatrix} = \frac{\mathrm{Det}\,(A)}{\mathrm{Det}\,(D)} \cdot \mathrm{Det}\left(I_p - A^{-1}BD^{-1}C\right). \tag{1.4}$$

Now, by lemma **??**, we have:

$$\mathrm{Det}\left(I_p - A^{-1}BD^{-1}C\right) \cdot \mathrm{Det}\left(I_q - D^{-1}CA^{-1}B\right) = 1$$
$$\Rightarrow \mathrm{Det}\left(I_p - A^{-1}BD^{-1}C\right) = \frac{1}{\mathrm{Det}\,(I_q - D^{-1}CA^{-1}B)}. \tag{1.5}$$

Replacing (1.5) in (1.4), we find

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}\,(A)}{\mathrm{Det}\,(D)\,\mathrm{Det}\,(I_q - D^{-1}CA^{-1}B)} = \frac{\mathrm{Det}\,(A)}{\mathrm{Det}\,(D - CA^{-1}B)}$$

□

Remark 1.102. At this point, it is interesting to compare the expression above with the expression of the traditional determinant.

Traditional determinant:

$$\mathrm{Det}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Det}\,(A)\,\mathrm{Det}\left(D - CA^{-1}B\right) = \mathrm{Det}\,(D)\,\mathrm{Det}\left(A - BD^{-1}C\right).$$

Chapter

2

Supersmooth functions, supermanifolds and integration

2.1 Grassmann algebra

In the previous chapter, we adopted a general graded commutative ring $\mathcal{A}$. From now on, the complex (real) Grassmann algebra will represent our $\mathcal{A}$.

Based on [16], we will define the complex Grassmann algebra e its conjugation.

Definition 2.1. (Complex Grassmann algebra) For each finite positive integer L, $\mathbb{C}_{S[L]}$ denotes the Grassmann algebra over $\mathbb{C}$ with L generators. That is, $\mathbb{C}_{S[L]}$ is the algebra over $\mathbb{C}$ with generators

$$1, \beta_{[1]}, \beta_{[2]}, \ldots, \beta_{[L]}$$

and relations

$$\begin{aligned} 1\beta_{[i]} &= \beta_{[i]} = \beta_{[i]}1 \qquad i = 1, \ldots, L \\ \beta_{[i]}\beta_{[j]} &= -\beta_{[j]}\beta_{[i]} \qquad i, j = 1, \ldots, L \end{aligned}$$

As a direct consequence, we have $\beta_{[i]}\beta_{[i]} = 0, \ \forall i$.

If $X \in \mathbb{C}_{S[L]}$, then:

$$X = \sum_{\lambda \in M_L} X_\lambda \beta_{[\lambda]}$$

where M_L is the set of all multi-indices (including the empty index $\emptyset$); $\beta_{[\emptyset]} = 1$; $\lambda \in M_L \Rightarrow \lambda = \lambda_1 \cdots \lambda_k$, with $1 \leq \lambda_1 < \cdots < \lambda_k \leq L$ and $\beta_{[\lambda]} = \beta_{[\lambda_1]} \cdots \beta_{[\lambda_k]}$. Furthermore, $X_\lambda \in \mathbb{C}$, $\forall \lambda \in M_L$.

We define the **even elements** of $\mathbb{C}_{S[L]}$ by:

$$\mathbb{C}_{S[L_0]} = \left\{ x | x \in \mathbb{C}_{S[L]},\ x = \sum_{\lambda \in M_{L_0}} X_\lambda \beta_{[\lambda]} \right\} \tag{2.1}$$

where M_{L_0} is the set of multi-indices with even numbers of indices. Now, we define the **odd elements** of $\mathbb{C}_{S[L]}$ by:

$$\mathbb{C}_{S[L_1]} = \left\{ x | x \in \mathbb{C}_{S[L]},\ x = \sum_{\lambda \in M_{L_1}} X_\lambda \beta_{[\lambda]} \right\} \tag{2.2}$$

where M_{L_1} is the set of multi-indices with odd numbers of indices.

Remark 2.2. This construction give to $\mathbb{C}_{S[L]}$ a structure of graded commutative ring, and the set of nilpotent elements is equal to:

$$\mathcal{N} = \sum_{\lambda \in (M_L \setminus \{\emptyset\})} X_\lambda \beta_{[\lambda]}$$

Thus, the body map $\mathbb{B} : \mathbb{C}_{S[L]} \to \mathbb{C} \simeq \frac{\mathbb{C}_{S[L]}}{\mathcal{N}}$ is equal to:

$$\mathbb{B}\left(\sum_{\lambda \in M_L} X_\lambda \beta_{[\lambda]} \right) = X_\emptyset.$$

Remark 2.3. The real Grassmann algebra is quite similar, with one difference: the coefficients X_λ are real, i.e, if $X \in \mathbb{R}_{S[L]}$, then:

$$X = \sum_{\lambda \in M_L} X_\lambda \beta_{[\lambda]}$$

with $X_\lambda \in \mathbb{R}$.

Remark 2.4. From now on, we will consider Grassmann algebras with infinite quantity of generators, i.e., with $L \to \infty$, and we will denote these specific Grassmann algebras by $\mathbb{C}_S$ and by $\mathbb{R}_S$.

2.2 Superspaces

In the next definition, we will use the notations of (2.1) and (2.2).

Definition 2.5. We define the $m|n$ dimension superspace $\mathbb{C}_{S[L]}^{m,n}$ by

$$\mathbb{C}_{S[L]}^{m,n} = \underbrace{\mathbb{C}_{S[L_0]} \times \cdots \times \mathbb{C}_{S[L_0]}}_{\text{m copies}} \times \underbrace{\mathbb{C}_{S[L_1]} \times \cdots \times \mathbb{C}_{S[L_1]}}_{\text{n copies}}$$

with m said to be the even dimension and n the odd dimension of the superspace.

A typical element of $\mathbb{C}_{S[L]}^{m,n}$ is denoted by $(x^1, \ldots, x^m, \xi^1 \ldots, \xi^n)$, or more briefly as (x, ξ).

$\mathbb{C}_S^{m,n}$ is found doing $L \to \infty$, and

$$\mathbb{C}_S^{m,n} = \underbrace{\mathbb{C}_{S_0} \times \cdots \times \mathbb{C}_{S_0}}_{\text{m copies}} \times \underbrace{\mathbb{C}_{S_1} \times \cdots \times \mathbb{C}_{S_1}}_{\text{n copies}}$$

A complex superspace $\mathbb{C}_S^{m,n}$ of $m|n$ dimension is naturally a real superspace $\mathbb{R}_S^{2m,2n}$ of $2m|2n$ dimension. Complexifying this superspace, we have:

Definition 2.6. On $\mathbb{R}_S^{2m,2n} \otimes \mathbb{C}_S$, we define:

(a) (Even holomorphic and antiholomorphic variables)

$$z_j = x_j + iy_j \qquad \overline{z}_j = x_j - iy_j$$

with x_j, y_j even variables and $j \in \{1, \ldots, m\}$.

(b) (Odd holomorphic and antiholomorphic variables)

$$\xi_\kappa = \eta_\kappa + i\zeta_\kappa \qquad \overline{\xi}_\kappa = \eta_\kappa - i\zeta_\kappa$$

with $\eta_\kappa, \zeta_\kappa$ odd variables and $\kappa \in \{1, \ldots, n\}$.

2.3 DeWitt topology

In this work, we will adopt the DeWitt topology.

Definition 2.7. A subset U of $\mathbb{C}_S^{m,n}$ is said to be open in the DeWitt topology on $\mathbb{C}_S^{m,n}$ if and only if there exists an open subset V of $\mathbb{C}^m$ such that

$$U = \mathbb{B}^{-1}(V).$$

Remark 2.8. The DeWitt topology is a non-Hausdorff topology.

2.4 Superholomorphic functions

By [20] and [16], we define:

Definition 2.9. Suppose that V is an open subset of $\mathbb{C}^m$ and U is a subset of $\mathbb{C}_s^{m,n}$ such that $\mathbb{B}(U) = V$. Then:

$$\begin{aligned} \widehat{\ }: \quad C^\infty(V,\mathbb{C}) &\longrightarrow C^\infty(U,\mathbb{C}_s) \\ f(\mathbb{B}(z)) &\longrightarrow \widehat{f(z)} \end{aligned}$$

The function $\widehat{f}$ is called the Grassmann analytic continuation of f.

Remark 2.10. An expression for $\widehat{f(z)}$ is found in [16].

Definition 2.11. Let U be an open set in the $\mathbb{C}_S$-vector space $\mathbb{C}_s^{m,n}$. Then $f : U \to \mathbb{C}_S$ is said to be a superholomorphic function on U if and only if there exists a collection $\{f_\mu | \mu \in M_n\}$ of $\mathbb{C}_S$ - valued functions which are holomorphic on $\mathbb{B}(U)$ such that

$$f(z^1, \cdots, z^m; \xi^1, \cdots, \xi^n) = \sum_{\mu \in M_n} \xi^\mu \cdot \widehat{f_\mu}(z^1, \cdots, z^m)$$

where $\mu \in M_n$ is a multi-index $\mu = \{\mu_{i_1}, \cdots, \mu_{i_k}\}$, with $\mu_{i_1} < \cdots < \mu_{i_k}$.

Let us generalize the definition 2.11 to a complexified space with holomorphic and anti-holomorphic variables.

Definition 2.12. Let U be an open set in a complexified space of $2m|2n$ dimension. Then $f : U \to \mathbb{C}_s$ is said to be a supersmooth function on U if and only if there exists a collection $\{f_{\mu,\lambda} | \mu \in M_n, \lambda \in N_n\}$ of $\mathbb{C}_s$ - valued functions which are smooth functions on $\mathbb{B}(U)$ such that

$$f(z^1, \cdots, z^m, \overline{z}^1, \cdots, \overline{z}^m; \xi^1, \cdots, \xi^n, \overline{\xi}^1, \cdots, \overline{\xi}^n) = \sum_{\mu \in M_n, \lambda \in N_n} \xi^\mu \cdot \overline{\xi}^{\ \lambda} \cdot \widehat{f_{\mu;\lambda}}(z^1, \cdots, z^m, \overline{z}^1 \cdots, \overline{z}^m)$$

where $\mu \in M_n$ is a multi-index $\mu = \{\mu_{i_1}, \cdots, \mu_{i_k}\}$, with $\mu_{i_1} < \cdots < \mu_{i_k}$, and $\lambda \in N_n$ is other multi-index $\lambda = \{\lambda_{j_1}, \cdots, \lambda_{j_l}\}$, with $\lambda_{j_1} < \cdots < \lambda_{j_l}$.

2.5 Partitions of unity

Theorem 2.13. Suppose that U is open in $\mathbb{C}_S^{m,n}$. Let $\{U_\alpha|\ \alpha \in \Lambda\}$ be a locally finite open cover of U. Then there exist $C^\infty(U_\alpha, \mathbb{C}_S)$ functions $\{f_\alpha|\ \alpha \in \Lambda\}$ with the support of each f_α contained in U_α such that

$$\sum_{\alpha\in\Lambda} f_\alpha = 1.$$

A collection of functions with these properties is said to be a partition of unity on U subordinate to $\{U_\alpha|\ \alpha \in \Lambda\}$.

Proof. Vide [16, pg 49]. □

2.6 Supermanifolds

Definition 2.14. Let M be a set, and let m and n be positive integers.

(i) An $m|n$ superholomorphic chart on M is a pair (V, ψ) where V is a subset of M and ψ is a bijective superholomorphic map from V onto an open subset of $\mathbb{C}_S^{m,n}$ (in the DeWitt topology);

(ii) An $m|n$ superholomorphic atlas on M is a collection of $m|n$ superholomorphic charts $\{(V_\alpha, \psi_\alpha)|\ \alpha \in \Lambda\}$ such that

a. $\bigcup_{\alpha\in\Lambda} V_\alpha = M$

b. for each α, β in Λ such that $V_\alpha \bigcap V_\beta \neq \emptyset$ the map

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta\left(V_\alpha \bigcap V_\beta\right) \to \psi_\alpha\left(V_\alpha \bigcap V_\beta\right)$$

is a bijective superholomorphic map $C^\infty\left(\psi_\beta\left(V_\alpha \bigcap V_\beta\right); \psi_\alpha\left(V_\alpha \bigcap V_\beta\right)\right)$;

(iii) An $m|n$ superholomorphic atlas $\{(V_\alpha, \psi_\alpha)|\ \alpha \in \Lambda\}$ on M which is not contained in any other such atlas on M is called a complete $m|n$ superholomorphic atlas on M.

Based on definition 2.14, we have the following important definition.

Definition 2.15. An $m|n$ complex (DeWitt) supermanifold consists of a set M together with a complete $m|n$ superholomorphic atlas on M.

Remark 2.16. The maps ψ_α are called coordinate maps and the sets V_α are called coordinate neighbourhoods.

Remark 2.17. To define a topology on M, it is necessary to require that each coordinate map is a homeomorphism of the corresponding coordinate neighbourhood onto its image.

Remark 2.18. Since the DeWitt topology of $\mathbb{C}_S^{m,n}$ is not Hausdorff, then the DeWitt topology of a supermanifold is also non-Hausdorff.

Remark 2.19. By definition 2.15 we note that the complex supermanifolds are modelled locally on $\mathbb{C}_S^{m,n}$. Furthermore, we conclude too that $\mathbb{C}_S^{m,n}$ is itself such a supermanifold, as is any open subset of $\mathbb{C}_S^{m,n}$.

The previous definition of supermanifold has the approach of differential geometry. There is too one another definition that is linked with the algebraic geometry perspective. Batchelor in [2] shown that the correspondence is so close that it is often unnecessary to state explicitly which approach is being used.

The algebro-geometric definition of a supermanifold was given independently in broadly equivalent form by Berezin and Leĭtes [28] and Kostant [29].

Definition 2.20 (Algebro-geometric definition)**.** A complex supermanifold is a ringed space $S = (X, \mathcal{O}_S)$, where X is a complex manifold and $\mathcal{O}_S$ is a sheaf of super commutative algebras over X locally isomorphic to an exterior algebra on the vector bundle.

Definition 2.21. The tangent bundle TS of $S = (X, \mathcal{O}_S)$ is defined by:

$$TS := Der(\mathcal{O}_S).$$

Remark 2.22. At this thesis, the differential geometry perspective is sufficient. But further material on the algebro-geometric approach may be found in [30, 31, 32].

2.7 Body map on supermanifolds

Theorem 2.23. Let M be a complex supermanifold with atlas $\{(V_\alpha, \psi_\alpha) \mid \alpha \in \Lambda\}$. Then

a. the relation $\simeq$ defined on M by $p \simeq q$ if and only if there exists $\alpha \in \Lambda$ such that both p and q lie in V_α and

$$\mathbb{B}\left(\psi_\alpha(p)\right) = \mathbb{B}\left(\psi_\alpha(q)\right)$$

is an equivalence relation.

b. The space $\mathbb{B}(M) = M/\simeq$ has the structure of a m-dimensional manifold with atlas $\{(V_{\emptyset\alpha}, \psi_{\emptyset\alpha}) \mid \alpha \in \Lambda\}$ where $V_{\emptyset\alpha} = \{[p] \mid p \in V_\alpha\}$ and

$$\begin{array}{rccl} \psi_{\emptyset\alpha} & : & V_{\emptyset\alpha} & \to & \mathbb{C}^m \\ & & [p] & \to & \mathbb{B}(\psi_\alpha(p)). \end{array}$$

(Here square brackets [ ] denote equivalence classes in M under $\simeq$)

Proof. Vide [16, pg 60]. □

Definition 2.24. The manifold $M/\simeq$ is called the body of M and denoted by $M_\emptyset$. The canonical projection of M onto $M_\emptyset$ is denoted by $\mathbb{B}$.

$$\mathbb{B}: \quad M_s^{m,n} \longrightarrow {M_\emptyset}^m$$

$$U \mapsto \mathbb{B}(U) = U_\emptyset$$

Example 2.25. Let S be a supermanifold on $\mathbb{C}_s^{m,n}$, then $X = \mathbb{B}(S)$ is a manifold on $\mathbb{C}^m$, and X is called **the body manifold associated to** S.

Example 2.26. Let $SB_\epsilon(p) \subset S$ be a superball centered at $p \in S$ in the supermanifold S. Then $B_\epsilon(\mathbb{B}(p)) = \mathbb{B}(SB_\epsilon(p))$ is a ball centered at $\mathbb{B}(p)$ of radius ϵ in the manifold X.

Remark 2.27. Certain aspects of a supermanifold are determined by its body. For instance, a supermanifold is compact if its body is compact, and simply connected if its body is simply connected, while the fundamental group of a supermanifold is simply the fundamental group of its body.

2.8 Superfunction's Derivation

Definition 2.28. Suppose that f is a supersmooth function. Then the derivative is defined as follows:

$$\frac{\partial}{\partial z_i} f(z,\overline{z};\xi,\overline{\xi}) = \sum_{\mu\in M_n,\lambda\in N_n} \xi^\mu \cdot \overline{\xi}^{\,\lambda} \cdot \widehat{\frac{\partial}{\partial z_i} f_{\mu;\lambda}}(z,\overline{z})$$

$$\frac{\partial}{\partial \overline{z}_i} f(z,\overline{z};\xi,\overline{\xi}) = \sum_{\mu\in M_n,\lambda\in N_n} \xi^\mu \cdot \overline{\xi}^{\,\lambda} \cdot \widehat{\frac{\partial}{\partial \overline{z}_i} f_{\mu;\lambda}}(z,\overline{z})$$

$$\frac{\partial}{\partial \xi_j} f(z,\overline{z};\xi,\overline{\xi}) = \sum_{\mu\in M_n,\lambda\in N_n} \rho_{j,\mu} \cdot \xi^{\mu/j} \cdot \overline{\xi}^{\,\lambda} \cdot \widehat{f_{\mu;\lambda}(z,\overline{z})}$$

where $\rho_{j,\mu} = \begin{cases} (-1)^{l-1} & \text{if } j = \mu_l \\ 0 & \text{otherwise} \end{cases}$ and $\mu/j = \begin{cases} \mu_1 \cdots \mu_{l-1}\mu_{l+1} \cdots \mu_k & \text{if } j = \mu_l \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial}{\partial \overline{\xi}_j} f(z, \overline{z}; \xi, \overline{\xi}) = \sum_{\mu \in M_n, \lambda \in N_n} (-1)^{L(\mu)} \rho_{j,\lambda} \cdot \xi^{\mu} \cdot (\overline{\xi})^{\lambda/j} \cdot \widehat{f_{\mu;\lambda}(z, \overline{z})}$$

where $\rho_{j,\lambda} = \begin{cases} (-1)^{l-1} & \text{if } j = \lambda_l \\ 0 & \text{otherwise} \end{cases}$ and $\lambda/j = \begin{cases} \lambda_1 \cdots \lambda_{l-1}\lambda_{l+1} \cdots \lambda_k & \text{if } j = \lambda_l \\ 0 & \text{otherwise} \end{cases}$

Remark 2.29. $L(\mu)$ is the length of the multi-index $\mu \in M_n$.

By [16] (Theorem 10.3.4) and by [17], page 160, we define the following:

Definition 2.30. Let M be a supermanifold of $m|n$ dimension, and let $f(z, \xi)$ be a superfunction, then:

$$\overline{\partial}\ \overline{f} = \underbrace{\sum_{i=1}^{m} d\overline{z}_i \frac{\partial \overline{f}}{\partial \overline{z}_i}}_{\overline{\partial}^E \overline{f}} + \underbrace{\sum_{j=1}^{n} d\overline{\xi}_j \frac{\partial \overline{f}}{\partial \overline{\xi}_j}}_{\overline{\partial}^O \overline{f}}.$$

Then

$$\overline{\partial}\ \overline{f} = \overline{\partial}^E \overline{f} + \overline{\partial}^O \overline{f}.$$

$\overline{\partial}^E \overline{f}$ is called the even derivation of $\overline{f}$ and $\overline{\partial}^O \overline{f}$ is called the odd derivation of $\overline{f}$.

Remark 2.31. Since $d^E = \partial^E + \overline{\partial}^E$ and $d^O = \partial^O + \overline{\partial}^O$, then if $f(z, \xi)$ is holomorphic, then $\overline{\partial}\ \overline{f} = d^E \overline{f} + d^O \overline{f}$.

Definition 2.32. Let U be an open set in the $\mathbb{C}_S$-vector space $\mathbb{C}_s^{m,n}$.

- We say that $f : U \to \mathbb{C}_S$ is an even supersmooth function on U if and only if there exists a collection $\{f_{\mu_E} | \mu_E \in M_{n_E}\}$ of $\mathbb{C}_S$ - valued functions, where M_{n_E} is the set of all even multi-indices, such that

$$f(z^1, \cdots, z^m; \xi^1, \cdots, \xi^n) = \sum_{\mu_E \in M_{n_E}} \xi^{\mu_E} \cdot \widehat{f_{\mu_E}}(z^1, \cdots, z^m)$$

where $\mu_E \in M_{n_E}$ is a even multi-index $\mu_E = \{\mu_{i_1}, \cdots, \mu_{i_{2k}}\}$, with $\mu_{i_1} < \cdots < \mu_{i_{2k}}$, for $k = 0, 1, 2, \ldots$.

- We say that $f : U \to \mathbb{C}_S$ is an odd supersmooth function on U if and only if there exists a collection $\{f_{\mu_O} | \mu_O \in M_{n_O}\}$ of $\mathbb{C}_S$ - valued functions, where M_{n_O} is the set of all odd multi-indices, such that

$$f(z^1, \cdots, z^m; \xi^1, \cdots, \xi^n) = \sum_{\mu_O \in M_{n_O}} \xi^{\mu_O} \cdot \widehat{f_{\mu_O}}(z^1, \cdots, z^m)$$

where $\mu_O \in M_{n_O}$ is a odd multi-index $\mu_O = \{\mu_{i_1}, \cdots, \mu_{i_{(2k+1)}}\}$, with $\mu_{i_1} < \cdots < \mu_{i_{(2k+1)}}$, for $k = 0, 1, 2, \ldots$.

Definition 2.33. Let $dX = dx_1 \wedge \cdots \wedge dx_n$ be a homogeneous n-form (i.e, each dx_j is homogeneous) and let f be a homogeneous superfunction, then the parity $\varepsilon(dXf)$ is given by the formula:

$$\varepsilon(dXf) = \varepsilon(dX) + \varepsilon(f) = \left(\sum_{i=1}^{n} \varepsilon(dx_i) \right) + \varepsilon(f).$$

Lemma 2.34. If f is an odd (even) superfunction, then $\overline{\partial}\ \overline{f}$ is an odd (even) 1-form.

Proof. Let $\overline{\partial}\ \overline{f} = \sum_{i=1}^{m} d\overline{z}_i \frac{\partial \overline{f}}{\partial \overline{z}_i} + \sum_{j=1}^{n} d\overline{\xi}_j \frac{\partial \overline{f}}{\partial \overline{\xi}_j}$.. Then if f is odd, we have that $\frac{\partial \overline{f}}{\partial \overline{z}_i}$ is odd, and $d\overline{z}_i \frac{\partial \overline{f}}{\partial \overline{z}_i}$ will be odd too. In the same way, if f is odd, $\frac{\partial \overline{f}}{\partial \overline{\xi}_j}$ will be even, and $d\overline{\xi}_j \frac{\partial \overline{f}}{\partial \overline{\xi}_j}$ will be odd. Therefore, $\overline{\partial}\ \overline{f}$ will be odd.

Now, to $\overline{\partial}\ \overline{f} = \sum_{i=1}^{m} d\overline{z}_i \frac{\partial \overline{f}}{\partial \overline{z}_i} + \sum_{j=1}^{n} d\overline{\xi}_j \frac{\partial \overline{f}}{\partial \overline{\xi}_j}$., if f is even, then $\frac{\partial \overline{f}}{\partial \overline{z}_i}$ will be even, and consequently $d\overline{z}_i \frac{\partial \overline{f}}{\partial \overline{z}_i}$ will be even too. Beyond that, $\frac{\partial \overline{f}}{\partial \overline{\xi}_j}$ will be odd, and consequently $d\overline{\xi}_j \frac{\partial \overline{f}}{\partial \overline{\xi}_j}$ will be even. Therefore, $\overline{\partial}\ \overline{f}$ will be even. □

Remark 2.35. By lemma above, we conclude that if f is even, then $\overline{\partial}^E \overline{f}$ and $\overline{\partial}^O \overline{f}$ are both even, and if f is odd, then $\overline{\partial}^E \overline{f}$ and $\overline{\partial}^O \overline{f}$ are both odd.

By [16] (Theorem 10.3.4), we have

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^p \alpha \wedge d(\beta),$$

with α being an p-form.

Therefore, by the above result, by [11] (page 17), and by proposition 1.75 , we get the following definition:

Definition 2.36. Let dX be an even or odd 1-form. Then, for the section $dX\overline{f}$, with f holomorphic, we have:

$$\overline{\partial}(dX\overline{f}) = \sum_{i=1}^{m}(-1)^{\langle\varepsilon(dX)|\varepsilon(d\overline{z}_i)\rangle} d\overline{z}_i \wedge dX\frac{\partial\overline{f}}{\partial\overline{z}_i} + \sum_{j=1}^{n}(-1)^{\left\langle\varepsilon(dX)|\varepsilon(d\bar{\xi}_j)\right\rangle} d\bar{\xi}_j \wedge dX\frac{\partial\overline{f}}{\partial\bar{\xi}_j}.$$

From definition 2.36, we deduce easily the following analogous formula:

$$\overline{\partial}(dX\overline{f}) = \sum_{i=1}^{m} -dX \wedge d\overline{z}_i\frac{\partial\overline{f}}{\partial\overline{z}_i} + \sum_{j=1}^{n} -dX \wedge d\bar{\xi}_j\frac{\partial\overline{f}}{\partial\bar{\xi}_j},$$

and

$$\overline{\partial}(dX\overline{f}) = -dX \wedge d^E\overline{f} - dX \wedge d^O\overline{f}. \tag{2.3}$$

2.9 Body map on superfunctions

Definition 2.37. Body map applied on superfunctions:

$$\begin{aligned} \mathbb{B}\colon\quad C^\infty(U,V) &\longrightarrow C^\infty(\mathbb{B}(U),\mathbb{B}(V)) \\ f &\mapsto \mathbb{B}(f) = f_\emptyset \end{aligned}$$

Therefore, the supersmooth function f become $f_\emptyset$, that is a classic smooth function applied on the body.

Example 2.38. To the superfunction $f(z,\overline{z},\xi,\overline{\xi})$, we have $\mathbb{B}(f(z,\overline{z},\xi,\overline{\xi})) = f_\emptyset\,(\mathbb{B}(z),\mathbb{B}(\overline{z}))$.

2.10 Body map on superforms

Definition 2.39. Body map applied on superforms:

$$\begin{aligned} \mathbb{B}\colon\quad A^{(p,q)|(r,s)} &\longrightarrow A^{p,q} \\ \eta &\mapsto \mathbb{B}(\eta) = \eta_\emptyset \end{aligned}$$

Example 2.40. Let $\eta \in A^{(p,q)|(r,s)}$ be $\eta = f\ dz^1 \wedge \cdots \wedge dz^p \wedge d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^q \wedge d\xi^1 \wedge \cdots \wedge d\xi^r \wedge d\overline{\xi}^1 \wedge \cdots \wedge d\overline{\xi}^s$. Then $\mathbb{B}(\eta) = \eta_\emptyset \in A^{p,q}$ is $\eta_\emptyset = f_\emptyset\ dz^1 \wedge \cdots \wedge dz^p \wedge d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^q$.

2.11 Integration on supermanifolds

This section is based on [16], and the main point is the new concept of the *Berezin integral*. We will begin defining the Berezin integral on a purely odd superspace $\mathbb{R}_S^{0,n}$ ($\mathbb{C}_S^{0,n}$). Next, we will define this integral on a superspace $\mathbb{R}_S^{m,n}$ ($\mathbb{C}_S^{m,n}$) and on a supermanifold M. Finally we will generalize the concept of the Berezin integral to even and odd holomorphic and anti-holomorphic variables on a complexified superspace.

The theory of integration on the purely odd superspace $\mathbb{R}_S^{0,n}$ ($\mathbb{C}_S^{0,n}$) is based on the following definition:

Definition 2.41. Suppose that f is a $C^\infty(\mathbb{R}_S^{0,n}, \mathbb{R}_S)$ function of $\mathbb{R}_S^{0,n}$ ($\mathbb{C}_S^{0,n}$) into $\mathbb{R}_S$ ($\mathbb{C}_S$) with

$$f(\xi^1, \ldots, \xi^n) = \xi^1 \cdots \xi^n f_{1\ldots n} + \text{ lower order terms.}$$

Then the Berezin integral of f is defined to be

$$\int \mathrm{d}^n\xi f(\xi^1, \ldots, \xi^n) = f_{1\ldots n}.$$

Now, let's define an integral over an open set U in $\mathbb{R}_S^{m,n}$ ($\mathbb{C}_S^{m,n}$) in terms of a Berezin integral over anticommuting variables together with an ordinary integral over the body $\mathbb{B}(U)$ of U.

Definition 2.42. Let U be open in $\mathbb{R}_S^{m,n}$ ($\mathbb{C}_S^{m,n}$) and $f : U \to \mathbb{R}_S$ ($\mathbb{C}_S$) be $C^\infty(U, \mathbb{R}_S)$. Then the integral of f over U is defined to be

$$\int_U \mathrm{d}^m x \mathrm{d}^n \xi \; f(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n) = \int_{\mathbb{B}(U)} \mathrm{d}^m x \; f_{1\ldots n}(x^1, \ldots, x^m).$$

where the integration over $\mathbb{B}(U)$ is evaluated as a standard Riemann integral.

The method of integration on $\mathbb{R}_S^{m,n}$ ($\mathbb{C}_S^{m,n}$) developed leads naturally to an integral of a Berezin density on a compact supermanifold. Like conventional manifolds, a partition of unity is used to sum the contribution from different coordinate patches.

Definition 2.43. Suppose that ω is a Berezin density on M and that the collection $\{(V_\alpha, f_\alpha) | \alpha \in \Gamma\}$ is a partition of unity on M where each V_α is a coordinate neighborhood with corresponding coordinate map ψ_α. Then the integral of ω over M is defined to be

$$\int_M \omega = \sum_{\alpha \in \Gamma} \int_{\psi_\alpha(V_\alpha)} \mathrm{d}^m x \mathrm{d}^n \xi \; \omega_\alpha(x;\xi) f_\alpha \circ \psi_\alpha^{-1}(x;\xi),$$

where ω_α is the local representative of ω in the chart (V_α, ψ_α).

Now, we will generalize the definitions 2.41, 2.42 and 2.43 using holomorphic and anti-holomorphic variables.

Definition 2.44. [Berezin integral of supersmooth functions with odd holomorphic and antiholomorphic variables]

From the expansion

$$f(\xi^1,\ldots,\xi^n,\overline{\xi}^1,\ldots,\overline{\xi}^n)=\xi^1\cdots\xi^n\cdot\overline{\xi}^1\cdots\overline{\xi}^n f_{1,\ldots,n;1,\ldots,n}+\text{ lower order terms.}$$

we have:

$$\int d^n\xi\ d^n\overline{\xi}\ f(z^1,\cdots,z^m,\overline{z}^1,\cdots,\overline{z}^m;\xi^1,\cdots,\xi^n,\overline{\xi}^1,\cdots,\overline{\xi}^n)=f_{1,\ldots,n;1,\ldots,n}.$$

Definition 2.45 (Integral over $U\subset\mathbb{C}_S^{m,n}$)**.**

$$\int_U d^mz\ d^m\overline{z}\ d^n\xi\ d^n\overline{\xi}\ f(z^1,\cdots,z^m,\overline{z}^1,\cdots,\overline{z}^m,\xi^1,\cdots,\xi^n,\overline{\xi}^1,\cdots,\overline{\xi}^n)=$$

$$=\int_{\mathbb{B}(U)} d^mz\ d^m\overline{z}\left(\underbrace{\int d^n\xi\ d^n\overline{\xi}\ f(z^1,\cdots,z^m,\overline{z}^1,\cdots,\overline{z}^m,\xi^1,\cdots,\xi^n,\overline{\xi}^1,\cdots,\overline{\xi}^n)}_{\text{Berezin integral}}\right)=$$

$$=\int_{\mathbb{B}(U)} d^mz\ d^m\overline{z}\ \ f_{1,\ldots,n;1,\ldots,n}(z^1,\ldots,z^m,\overline{z}^1,\ldots,\overline{z}^m).$$

Definition 2.46. Suppose that ω is a Berezin density, i.e., ω is a volume form on the compact complex supermanifold S, and suppose that the collection $\{(V_\alpha,f_\alpha)|\alpha\in\Gamma\}$ is a partition of unity (that exists because the supermanifold is compact) on S where each V_α is a coordinate neighbourhood with corresponding coordinate map ψ_α. Then the integral of ω over S is defined to be

$$\int_S\omega=\sum_{\alpha\in\Gamma}\int_{\psi_\alpha(V_\alpha)} d^mz\ d^m\overline{z}\ d^n\xi\ d^n\overline{\xi}\quad[\omega\circ\psi_\alpha^{-1}](z,\overline{z};\xi,\overline{\xi})\ [f_\alpha\circ\psi_\alpha^{-1}](z,\overline{z};\xi,\overline{\xi}).$$

Chapter 3

Localization on supermanifolds

3.1 Definitions

By [16], page 61, we define:

Definition 3.1. The supermanifold S is called *compact* if $X = \mathbb{B}(S)$ is compact.

Based on [8], [16], [18] [21], [27], we define:

Definition 3.2. Let S be a supermanifold of dimension $m|n$ and let V, in local coordinates, be equal to the supervector field $V = \sum_{i=1}^{m} f_i(z,\xi)\frac{\partial}{\partial z_i} + \sum_{j=1}^{n} g_j(z,\xi)\frac{\partial}{\partial \xi_j}$ on S.

(i) We say that V is an even vector field if, for all local coordinates, f is an even function and g is an odd function. Therefore, expanding the homogeneous functions (see definition 2.32), we have:

$$V = \sum_{i=1}^{m}[\widehat{f_i(z)}_\emptyset + \xi_1\xi_2\widehat{f_i(z)}_{12} + \cdots]\frac{\partial}{\partial z_i} + \sum_{j=1}^{n}[\xi_1\widehat{g_j(z)}_1 + \xi_2\widehat{g_j(z)}_2 + \cdots]\frac{\partial}{\partial \xi_j}$$

(ii) We say that V is an odd vector field if, for all local coordinates, f is an odd function and g is an even function. Therefore, expanding the homogeneous functions (see definition 2.32), we have:

$$V = \sum_{i=1}^{m}[\xi_1\widehat{f_i(z)}_1 + \xi_2\widehat{f_i(z)}_2 + \cdots]\frac{\partial}{\partial z_i} + \sum_{j=1}^{n}[\widehat{g_j(z)}_\emptyset + \xi_1\xi_2\widehat{g_j(z)}_{12} + \cdots]\frac{\partial}{\partial \xi_j}$$

Definition 3.3. Let S be a supermanifold of $m|n$ dimension. Then we define the form

$\eta \in \oplus A^{(p,q)|(r,s)}$ on S as

$$\eta = \widehat{\eta}_{(\emptyset;\emptyset)} + \xi_1\widehat{\eta}_{(1;\emptyset)} + \xi_2\widehat{\eta}_{(2;\emptyset)} + \cdots + \xi_{\gamma_1}\xi_{\gamma_2}\ldots\xi_{\gamma_\kappa}\widehat{\eta}_{(\gamma_1\gamma_2\cdots\gamma_\kappa;\emptyset)} + \overline{\xi}_1\widehat{\eta}_{(\emptyset;1)} + \overline{\xi}_2\widehat{\eta}_{(\emptyset;2)} + \cdots + \\ + \cdots + \xi_{\gamma_1}\ldots\xi_{\gamma_{\kappa_1}}\overline{\xi}_{\mu_1}\ldots\overline{\xi}_{\mu_{\kappa_2}}\widehat{\eta}_{(\gamma_1\cdots\gamma_{\kappa_1};\mu_1\cdots\mu_{\kappa_2})} + \cdots + \xi_1\ldots\xi_n\overline{\xi}_1\ldots\overline{\xi}_n\widehat{\eta}_{(1\cdots n;1\cdots n)}, \quad (3.1)$$

where each term $\widehat{\eta}_{(\alpha;\beta)}$ is given by

$$\widehat{\eta}_{(\alpha;\beta)} = \widehat{\eta}^{(0,0)|(0,0)}_{(\alpha;\beta)} + dz_1\ \widehat{\eta}^{(1,0)|(0,0)}_{(\alpha;\beta)} + dz_1 \wedge d\overline{z}_1\ \widehat{\eta}^{(1,1)|(0,0)}_{(\alpha;\beta)} + \cdots \\ \cdots + dz_1 \wedge \cdots \wedge dz_m \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_m \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n\ \widehat{\eta}^{(m,m)|(n,n)}_{(\alpha;\beta)} \quad (3.2)$$

To unify the notation, we define:

Definition 3.4. Let $\eta \in \oplus A^{(p,q)|(r,s)}$ be a form on S. To simplify, we will use the following notations:

$$(i) \quad \int_X \eta_{(1\cdots n;1\cdots n)} \quad := \quad \int_X \bigoplus_{(r,s)} dz_1 \wedge \cdots \wedge dz_r \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_s\ \eta^{(r,s)|(n,n)}_{(1\cdots n;1\cdots n)}$$

$$(ii) \quad \int_X \eta^{(r,s)|(n,n)}_{(1\cdots n;1\cdots n)} \quad := \quad \int_X dz_1 \wedge \cdots \wedge dz_r \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_s\ \eta^{(r,s)|(n,n)}_{(1\cdots n;1\cdots n)}$$

where $\eta^{(r,s)|(n,n)}_{(1\cdots n;1\cdots n)}$ are superfunctions.

Remark 3.5. Therefore, based on the definitions 2.45 (Berezin integral) and 3.4, we have:

$$\int_S \eta \ = \ \int_X \eta_{(1\cdots n;1\cdots n)} \ = \ \int_X \bigoplus_{(r,s)} dz_1 \wedge \cdots \wedge dz_r \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_s\ \eta^{(r,s)|(n,n)}_{(1\cdots n;1\cdots n)} \quad (3.3)$$

3.2 Localization formula for even vectors

Lemma 3.6. Let S be a compact complex supermanifold of dimension $m|n$. Then, for all $\omega \in \oplus A^{(p,q)|(r,s)}$, we have:

$$\int_S \overline{\partial}\omega = 0$$

Proof. It's sufficient to consider the following two cases:

(i) $\omega \in A^{(m,m)|(n,n-1)}$, then $\int_S \overline{\partial}\omega = 0$.

Indeed, based on [11], [16] and [17], we have:

$$\overline{\partial}\omega = \underbrace{\sum_i d\overline{z}^i \wedge \frac{\partial\omega}{\partial\overline{z}^i}}_{=0} + \sum_j d\overline{\xi}^j \wedge \frac{\partial\omega}{\partial\overline{\xi}^j} = \sum_j d\overline{\xi}^j \wedge \frac{\partial\omega}{\partial\overline{\xi}^j}.$$

Since, $\forall j$, the expansion of $d\overline{\xi}^j \wedge \frac{\partial\omega}{\partial\overline{\xi}^j}$ in powers of ξ and $\overline{\xi}$ has no top term, then, by definition 2.45 of the Berezin integral, we have:

$$\int_S \overline{\partial}\omega = \int_S \sum_j d\overline{\xi}^j \wedge \frac{\partial\omega}{\partial\overline{\xi}^j} = \sum_j \int_S d\overline{\xi}^j \wedge \frac{\partial\omega}{\partial\overline{\xi}^j} = 0$$

(ii) $\omega \in A^{(m,m-1)|(n,n)}$, then $\int_S \overline{\partial}\omega = 0$.

First, observe that $\int_S \partial\omega = 0$ because

$$\partial\omega = \underbrace{\sum_i dz^i \wedge \frac{\partial\omega}{\partial z^i}}_{=0} + \sum_j d\xi^j \wedge \frac{\partial\omega}{\partial\xi^j} = \sum_j d\xi^j \wedge \frac{\partial\omega}{\partial\xi^j}$$

and since $\sum_j d\xi^j \wedge \frac{\partial\omega}{\partial\xi^j} \in A^{(m,m-1)|(n+1,n)}$, then $\int_S \sum_j d\xi^j \wedge \frac{\partial\omega}{\partial\xi^j} = 0$ and therefore $\int_S \partial\omega = 0$.

Now, without loss of generality, consider ω as following:

$$\omega = dz^1 \wedge \cdots \wedge \widehat{d\overline{z}^i} \wedge \cdots \wedge d\overline{z}^m \wedge d\xi^1 \wedge \cdots \wedge d\overline{\xi}^n [\xi^1 \ldots \xi^n \cdot \overline{\xi}^{\,1} \ldots \overline{\xi}^{\,n} \omega_{(1,\ldots,n;1,\ldots,n)} + (\text{lower terms})].$$

Then, by the Berezin integral and by stokes, we have:

$$\int_S \overline{\partial}\omega = \int_S (\partial + \overline{\partial})\omega = \int_S d\omega =$$

$$\int_{X=\mathbb{B}(S)} d^m z\, d^{m-1}\overline{z} \int d^n\xi\, d^n\overline{\xi}\;\; d\,[\xi^1 \ldots \xi^n \cdot \overline{\xi}^{\,1} \ldots \overline{\xi}^{\,n} \omega_{(1,\ldots,n;1,\ldots,n)} + (\text{lower terms})] =$$

$$\int_X d^m z\, d^{m-1}\overline{z}\;\; d\omega_{(1,\ldots,n;1,\ldots,n)} \overset{\text{(by stokes)}}{=} 0$$

□

Remark 3.7. From now on, we will use the notations of definitions 2.37 and 2.39.

Lemma 3.8. Let S be a supermanifold of dimension $m|n$ and let V be an even holomorphic vector field on S. If $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, then, for all $\omega \in \oplus A^{(p,q)|(r,s)}$ and $t > 0$, we have:

$$\int_S \eta = \int_S \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$$

Proof.

$$\frac{\partial}{\partial\kappa}\int_S \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta = \frac{\partial}{\partial\kappa}\int_X \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)} = \\ -\int_X (\overline{\partial} + i_{V_\emptyset})\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}.$$

Now, observe that, since $(\overline{\partial} + i_{V_\emptyset})^2\omega_\emptyset = 0$ (see [22]), we have:

$$(\overline{\partial} + i_{V_\emptyset}) \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} = \sum_{L=0}^{m}(\overline{\partial} + i_{V_\emptyset})\frac{\left(-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right)^L}{L!} = 0. \tag{3.4}$$

Since $(\overline{\partial} + i_V)(\eta) = 0$, then:

$$(\overline{\partial} + i_{V_\emptyset})\eta_{(1,\ldots,n;1,\ldots,n)} = 0 \tag{3.5}$$

Therefore, by (3.4) and (3.5), we have:

$$-\int_X (\overline{\partial} + i_{V_\emptyset})\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)} = \\ -\int_X (\overline{\partial} + i_{V_\emptyset})(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)})$$

Then:

$$\int_X (\overline{\partial} + i_{V_\emptyset})(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}) = \\ \int_X \overline{\partial}(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}) + \int_X i_{V_\emptyset}(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}).$$

By lemma 3.6, we have $\int_X \overline{\partial}(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta) = 0$ and as the contraction $i_{V_\emptyset}$ promotes the lost of top form in the integral, then $\int_X i_{V_\emptyset}(\omega_\emptyset \cdot \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta) = 0$.

Therefore $\frac{\partial}{\partial \kappa} \int_X \exp\left\{-\kappa(\overline{\partial} + i_{V_\emptyset})\omega_\emptyset\right\} \cdot \eta_{(1,\dots,n;1,\dots,n)} = 0$. Then, we have:

$$\int_S \eta = \int_X \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta_{(1,\dots,n;1,\dots,n)} = \int_S \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$$

□

Definition 3.9. Let S be a supermanifold of $m|n$ dimension, let V be an even supervector on S and let $Sing(V_\emptyset)$ be the set of points $\mathbb{B}(p)$ on X such that $V_\emptyset(\mathbb{B}(p)) = 0$. Then we define the set $Sing(V)$ as

$$Sing(V) := \{p \in S \mid \mathbb{B}(p) \in Sing(V_\emptyset)\}$$

$Sing(V)$ is called singular set of V on S and $\forall p \in Sing(V)$ is called singular point or singularity of V on S.

Definition 3.10. We say that $Sing(V)$ is a set of isolated singularities of V if $Sing(V_\emptyset)$ is a set of isolated singularities of $V_\emptyset$.

Definition 3.11. Let V be an even supervector on S. We say that $p \in Sing(V)$ is a non-degenerate singularity of V if $\mathbb{B}\left[Ber(V)\right](\mathbb{B}(p))$ exists and $\mathbb{B}\left[Ber(V)\right](\mathbb{B}(p)) \neq 0$.

Let V be an even supervector with isolated singularities . Then choose, for each singularity p_i, superballs $SB_\epsilon(p_i)$ (with $B_\epsilon(\mathbb{B}(p_i)) = \mathbb{B}(SB_\epsilon(p_i)))$ such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) = \emptyset$, for each $\mathbb{B}(p_i) \neq \mathbb{B}(p_j)$, with $p_i, p_j \in Sing(V)$.

Remark 3.12. If $\mathbb{B}(p_i) = \mathbb{B}(p_j)$, then $SB_\epsilon(p_i) = SB_\epsilon(p_j)$ because we are working in the DeWitt topology (see definition 2.7), and, therefore, any superball cover all the odd part of the supermanifold.

For each $SB_\epsilon(p_i)$, consider, in local coordinates, the following even vector field

$$V = \sum_{j=1}^{m} f_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} g_j \frac{\partial}{\partial \xi_j} \tag{3.6}$$

with all f_j being even holomorphic functions and all g_j being odd holomorphic functions.

Now, consider the form in $SB_\epsilon(p_i)$:

$$\widetilde{\omega}_i = \sum dz_j \overline{f}_j + \sum d\xi_j \overline{g}_j \tag{3.7}$$

Observe that each $\widetilde{\omega}_i$ is defined only locally, because the functions f_j, g_j are holomorphic on the compact supermanifold S. If they were defined globally, then they should be constant functions ([19], [12]). We will use the partition of unity (see theorem 2.13) to define a global form ω on S.

With an appropriate partition of unity, we will construct a global form ω such that

$$\omega_{|_{SB_\epsilon(p_i)}} = \widetilde{\omega}_i, \ \forall i.$$

Definition 3.13. Let S be a compact supermanifold of dimension $m|n$, and let $SB_\epsilon(p_i)$ be superballs in S. Now, let $\mathfrak{U} = \left(S - \cup_i \overline{SB}_\epsilon(p_i)\right) \bigcup (\cup_i SB_{2\epsilon}(p_i))$ be an open cover of S, such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) = \emptyset$, with $p_i, p_j \in Sing(V)$, let ω^0 be a form defined on $\left(S - \cup_i \overline{SB}_\epsilon(p_i)\right)$ and let $\{\rho_i\}$ be an unity partition subordinated to $\mathfrak{U}$. Then we define ω as the global form given by:

$$\omega = \rho_0 \omega^0 + \sum_i \rho_i \widetilde{\omega}_i. \tag{3.8}$$

To complete the definition, we need to construct the form ω^0. Let $\mathfrak{V} = \bigcup_j V_j$ be an open cover of $\overline{S - \cup_i \overline{SB}_\epsilon(p_i)}$, where, for each V_j, we have $V = \sum_{i=1}^m f_i^j \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i^j \frac{\partial}{\partial \xi_i}$, and let $\{\mu\}$ be an unit partition subordinated to $\mathfrak{V}$. Then, we construct ω^0 as follows:

$$\omega^0 = \sum_j \mu_j \left(\sum dz_i \overline{f}_i^j + \sum d\xi_i \overline{g}_i^j \right). \tag{3.9}$$

Remark 3.14. The operation $S - \cup_i \overline{SB}_\epsilon(p_i)$ is defined as follows:

$$S - \cup_i \overline{SB}_\epsilon(p_i) := \mathbb{B}^{-1}\left(X - \cup_i \overline{B}_\epsilon\left(\mathbb{B}(p_i)\right)\right)$$

where $X = \mathbb{B}(S)$.

Now, we will demonstrate the theorem of residues to an even holomorphic supervector field V on S.

Theorem 3.15. If the even holomorphic vector field V is a supervector without singularities on S, then for any $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, we have:

$$\int_S \eta = 0.$$

Proof. Choose any hermitian metric on $TS_\emptyset$ and let $\omega_\emptyset$ be the dual 1-form to the vector

$V_\emptyset = \mathbb{B}(V)$ via this metric. Then $i_{V_\emptyset}(\omega_\emptyset) = |V_\emptyset|^2$. Since V has no zeros on S and $X = \mathbb{B}(S)$ is compact, then $\exists \delta > 0$ such that $|V_\emptyset|^2 > \delta$ on X. So, by Lemma 3.8, we have:

$$\left|\int_S \eta\right| = \left|\int_S \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta\right| = \left|\int_X \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{|V_\emptyset|^2}{t}\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}\right| =$$
$$\left|\int_X \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t}\right\} \exp\left\{-\frac{|V_\emptyset|^2}{t}\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}\right| \leq \int_X \left| \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t}\right\} \exp\left\{-\frac{|V_\emptyset|^2}{t}\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}\right|$$
$$\leq \int_X \left| \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t}\right\} \cdot \eta_{(1,\ldots,n;1,\ldots,n)}\right| e^{-\delta/t} \leq \sum_{\alpha=0}^{m} \frac{1}{\alpha!} \int_X \left| \left(-\frac{\overline{\partial}\omega_\emptyset}{t}\right)^\alpha \cdot \eta^{(m-\alpha,m-\alpha)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}\right| e^{-\delta/t} \leq$$
$$C \cdot e^{-\delta/t}\left(\frac{1}{t^m} + \frac{1}{t^{m-1}} + \cdots + 1\right) Vol(X)$$

where $C > 0$ is a constant that limit superiorly all continuous functions in the integrals (remember that X is a compact manifold). Now, taking $t \to 0$, we get the result. □

Theorem 3.16. Let V be an even holomorphic supervector field on S with isolated singularities $p_i \in Sing(V)$, then for any $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, we have:

$$\int_S \eta = \sum_i Res_{p_i}(V, \eta)$$

where

$$Res_{p_i}(V, \eta) = \lim_{t\to 0} \int_{SB_\epsilon(p_i)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$$

Proof. By Lemma 3.8, we have:

$$\int_S \eta = \int_S \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$$

Therefore, for $p_i, p_j \in Sing(V)$ such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) = \emptyset$, we have:

$$\int_S \eta = \underbrace{\int_{S-\left(\bigcup_i SB_\epsilon(p_i)\right)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta}_{=0 \text{ (by theorem 3.15)}} + \sum_i \int_{SB_\epsilon(p_i)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$$

Then:

$$\int_S \eta = \sum_i \int_{SB_\epsilon(p_i)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta.$$

Thus:

$$\int_S \eta = \lim_{t\to 0} \sum_i \int_{SB_\epsilon(p_i)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta.$$

Therefore, as $Res_{p_i}(V,\eta) = \lim_{t\to 0}\int_{SB_\epsilon(p_i)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\}\cdot\eta$, we finally have:

$$\int_S \eta = \sum_i Res_{p_i}(V,\eta)$$

□

3.3 Determination of residues for even vectors

In this section, we will determine, on some hypotheses, a formula to calculate the residues $Res_{p_i}(V,\eta)$. Let's start with two lemmas.

Lemma 3.17. Let V be a holomorphic even vector, with local coordinates equal to $V = \sum_{i=1}^m f_i\frac{\partial}{\partial z_i} + \sum_{j=1}^n g_j\frac{\partial}{\partial \xi_j}$, where f_i are holomorphic even functions and g_j are holomorphic odd functions. Then, given $\omega = \sum_{i=1}^m dz_i\overline{f}_i + \sum_{j=1}^n d\xi_j\overline{g}_j$, we have:

$$\left(\overline{\partial}\omega_\emptyset\right)^m = (-1)^{\frac{m(m+1)}{2}}\frac{m!}{det\,(JV_\emptyset)}df_{\emptyset_1}\wedge\cdots\wedge df_{\emptyset_m}\wedge d\overline{f}_{\emptyset_1}\wedge\cdots\wedge d\overline{f}_{\emptyset_m}.$$

Proof.

$$\omega_\emptyset = \sum_{i=1}^m dz_i\overline{f}_{\emptyset_i}.$$

$$\overline{\partial}\omega_\emptyset = \sum_{i=1}^m -dz_i\wedge d\overline{f}_{\emptyset_i} = \sum_{i=1}^m d\overline{f}_{\emptyset_i}\wedge dz_i.$$

$$\left(\overline{\partial}\omega_\emptyset\right)^m = \left(\sum_{i=1}^m d\overline{f}_{\emptyset_i}\wedge dz_i\right)^m = m!\; d\overline{f}_{\emptyset_1}\wedge dz_1\wedge\cdots\wedge d\overline{f}_{\emptyset_m}\wedge dz_m =$$

$$m!(-1)^{\frac{m(m+1)}{2}}dz_1\wedge\cdots\wedge dz_m\wedge d\overline{f}_{\emptyset_1}\wedge\cdots\wedge d\overline{f}_{\emptyset_m}.$$

Therefore:

$$\left(\overline{\partial}\omega_\emptyset\right)^m = (-1)^{\frac{m(m+1)}{2}}\frac{m!}{det\,(JV_\emptyset)}df_{\emptyset_1}\wedge\cdots\wedge df_{\emptyset_m}\wedge d\overline{f}_{\emptyset_1}\wedge\cdots\wedge d\overline{f}_{\emptyset_m}.$$

□

Lemma 3.18. Let $V = \sum_{i=1}^m f_i\frac{\partial}{\partial z_i} + \sum_{j=1}^n g_j\frac{\partial}{\partial \xi_j}$ be an even supervector field, in local coordinates, on the complex supermanifold S, and let $V_\emptyset = \mathbb{B}(V)$ be the vector field on X associated to V. If $\det\left[\left(\mathbb{B}\left(\frac{\partial g_k}{\partial \xi_l}(p_j)\right)\right)_{n\times n}\right] \neq 0$ for some point $p_j \in S$, then:

$$\mathbb{B}(\mathrm{Ber}V)(\mathbb{B}(p_j)) = \frac{\det[JV_\emptyset]}{\det(\mathbb{B}(D))}(\mathbb{B}(p_j)).$$

where

$$Ber(V)(z;\xi) = \mathrm{sdet}\begin{pmatrix} \frac{\partial f_i}{\partial z_j} & \frac{\partial g_k}{\partial z_j} \\ \frac{\partial f_i}{\partial \xi_l} & \frac{\partial g_k}{\partial \xi_l} \end{pmatrix}, \quad JV_{\emptyset} = \left(\mathbb{B}(\tfrac{\partial f_i}{\partial z_j})\right) \quad and \quad D = \left(\tfrac{\partial g_k}{\partial \xi_l}\right).$$

Proof. Consider $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with:

$$A = \left(\tfrac{\partial f_i}{\partial z_j}\right), \quad B = \left(\tfrac{\partial g_k}{\partial z_j}\right), \quad C = \left(\tfrac{\partial f_i}{\partial \xi_l}\right), \quad D = \left(\tfrac{\partial g_k}{\partial \xi_l}\right).$$

Since V is an even vector, we have that f is an even function and g an odd function. Therefore

$$f_i(z^1, \cdots, z^m; \xi^1, \cdots, \xi^n) = \sum_{\mu \in M_n} \xi^{\mu} \cdot \widehat{f^i_{\mu}}(z^1, \cdots, z^m)$$

with $L(\mu) =$ *even natural number* (where $L(\mu)$ represents the quantity of indices on the multi-index μ) and

$$g_k(z^1, \cdots, z^m; \xi^1, \cdots, \xi^n) = \sum_{\lambda \in M_n} \xi^{\lambda} \cdot \widehat{g^k_{\lambda}}(z^1, \cdots, z^m)$$

with $L(\lambda) =$ *odd natural number*. With this facts in hands we conclude that the matrices A and D are composed for even terms, while B and C for odd terms.

Since $\det(\mathbb{B}(D))(p_j) \neq 0$, then by definition 1.97, we have:

$$\mathrm{Ber}(V)(p_j) = \left[\det(A - BD^{-1}C) \cdot (\det(D))^{-1}\right](p_j).$$

Once that $BD^{-1}C$ is a matrix with nilpotent terms (B and C are odd matrices), then $\mathbb{B}(BD^{-1}C) = 0$ and, as direct consequence of the definitions 1.83 and 1.88, we have:

$$\mathbb{B}(\mathrm{Ber}(V))(\mathbb{B}(p_j)) = \mathbb{B}[\det(A - BD^{-1}C) \cdot (\det(D))^{-1}] = \mathbb{B}[\det(A - BD^{-1}C)] \cdot \mathbb{B}[(\det(D))^{-1}] =$$

$$= \frac{\det(\mathbb{B}(A - BD^{-1}C))}{\det(\mathbb{B}(D))} = \frac{\det(\mathbb{B}(A) - \mathbb{B}(BD^{-1}C))}{\det(\mathbb{B}(D))} = \frac{\det(\mathbb{B}(A))}{\det(\mathbb{B}(D))} = \frac{\det[JV_{\emptyset}]}{\det(\mathbb{B}(D))}(\mathbb{B}(p_j)).$$

□

Remark 3.19. By lemma 3.18, we have $\det(JV_\emptyset) = \mathbb{B}(\mathrm{Ber}(V)) \cdot \det(\mathbb{B}(D))$. Therefore:

$$\det(JV_\emptyset) \simeq \mathrm{Ber}(V) \cdot \det(D)$$

In other words, $\det(JV_\emptyset)$ belongs the same equivalence class of $\mathrm{Ber}(V) \cdot \det(D)$.

Theorem 3.20. Let V be an even holomorphic vector field with a non-degenerate isolated singularity $p_j \in Sing(V)$, let S be a compact complex supermanifold with $m|n$ dimension and let $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$. Then, we have:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^m \left[\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j)$$

where

$$Ber(V)(z;\xi) = \mathrm{sdet}\begin{pmatrix} \frac{\partial f_i}{\partial z_j} & \frac{\partial g_k}{\partial z_j} \\ \frac{\partial f_i}{\partial \xi_l} & \frac{\partial g_k}{\partial \xi_l} \end{pmatrix} \quad and \quad D = \left(\frac{\partial g_k}{\partial \xi_l}\right).$$

Proof. Since $Res_{p_j}(V,\eta) = \lim\limits_{t\to 0} \int_{SB_\epsilon(p_j)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta$, then it's sufficient to show that:

$$\lim_{t\to 0} \int_{SB_\epsilon(p_j)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta = \left(\frac{2\pi}{i}\right)^m \left[\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j).$$

with $p_j \in Sing(V)$.

Then, we have:

$$\int_{SB_\epsilon(p_j)} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta =$$

$$\int_{B_\epsilon(\mathbb{B}(p_j))} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t} - \frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta_{1,\ldots,n;1,\ldots,n} =$$

$$\int_{B_\epsilon(\mathbb{B}(p_j))} \exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t}\right\} \cdot \exp\left\{-\frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\} \cdot \eta_{1,\ldots,n;1,\ldots,n}$$

Based in (3.6) and (3.7), we adopt the notation: $i_{V_\emptyset}(\omega_\emptyset) = \langle V_\emptyset, V_\emptyset\rangle$.

Therefore (by the notation (ii) of the definition 3.4), we have:

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \int_{B_\epsilon(\mathbb{B}(p_j))} \exp\left\{-\left\langle \frac{V_\emptyset}{\sqrt{t}}, \frac{V_\emptyset}{\sqrt{t}}\right\rangle\right\} \cdot \left(\frac{\overline{\partial}\omega_\emptyset}{t}\right)^k \cdot \eta^{(m-k,m-k)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}$$

Now, linearizing the functions near the singularity ([22], [23]) and doing the change of variable $z \mapsto z\sqrt{t}$, we have:

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \int_{B_{\epsilon/\sqrt{t}}\ (\mathbb{B}(p_j))} \exp\left\{-\langle V_\emptyset, V_\emptyset\rangle\right\} \cdot \frac{(\overline{\partial}\omega_\emptyset)^k}{(\sqrt{t})^{2k}} \cdot \eta^{(m-k,m-k)|(n,n)}_{(1,\ldots,n;1,\ldots,n)} \det \begin{pmatrix} \sqrt{t} & & 0 \\ & \ddots & \\ 0 & & \sqrt{t} \end{pmatrix} =$$

$$= \sum_{k=0}^{m} \frac{(-1)^k}{k!} \int_{B_{\epsilon/\sqrt{t}}\ (\mathbb{B}(p_j))} \exp\left\{-\langle V_\emptyset, V_\emptyset\rangle\right\} \cdot (\overline{\partial}\omega_\emptyset)^k \cdot \eta^{(m-k,m-k)|(n,n)}_{(1,\ldots,n;1,\ldots,n)} (\sqrt{t})^{2m-2k}$$

Then, when $t \to 0$, we have:

$$\frac{(-1)^m}{m!} \int_{\mathbb{C}^m} \exp\left\{-\langle V_\emptyset, V_\emptyset\rangle\right\} \cdot \eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)} \cdot (\overline{\partial}\omega_\emptyset)^m$$

Now, by the lemma 3.17:

$$\begin{aligned}
&(-1)^m(-1)^{\frac{m(m+1)}{2}} \frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{det\,(JV_\emptyset)} (p_j) \times \\
&\int_{\mathbb{C}^m} \exp\left\{-\sum_{i=1}^{m} f_{\emptyset_i}\overline{f}_{\emptyset_i}\right\} df_{\emptyset_1} \wedge \cdots \wedge df_{\emptyset_m} \wedge d\overline{f}_{\emptyset_1} \wedge \cdots \wedge d\overline{f}_{\emptyset_m} = \\
&\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{det\,(JV_\emptyset)} (p_j) \int_{\mathbb{C}^m} \exp\left\{-\sum_{i=1}^{m} f_{\emptyset_i}\overline{f}_{\emptyset_i}\right\} df_{\emptyset_1} \wedge d\overline{f}_{\emptyset_1} \wedge \cdots \wedge df_{\emptyset_m} \wedge d\overline{f}_{\emptyset_m} = \\
&\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{det\,(JV_\emptyset)} (p_j) \cdot \int_{\mathbb{R}^{2m}} \exp\left\{-x_1^2 - y_1^2 - \cdots - x_m^2 - y_m^2\right\} \times \\
&det \begin{pmatrix} \frac{\partial f_{\emptyset_1}}{\partial x_1} & \frac{\partial f_{\emptyset_1}}{\partial y_1} & & & \\ \frac{\partial \overline{f}_{\emptyset_1}}{\partial x_1} & \frac{\partial \overline{f}_{\emptyset_1}}{\partial y_1} & & & \\ & & \ddots & & \\ & & & \frac{\partial f_{\emptyset_m}}{\partial x_m} & \frac{\partial f_{\emptyset_m}}{\partial y_m} \\ & & & \frac{\partial \overline{f}_{\emptyset_m}}{\partial x_m} & \frac{\partial \overline{f}_{\emptyset_m}}{\partial y_m} \end{pmatrix} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_m \wedge dy_m
\end{aligned} \tag{3.10}$$

But since

$$det\begin{pmatrix} \frac{\partial f_{\emptyset_1}}{\partial x_1} & \frac{\partial f_{\emptyset_1}}{\partial y_1} & & & \\ \frac{\partial \overline{f}_{\emptyset_1}}{\partial x_1} & \frac{\partial \overline{f}_{\emptyset_1}}{\partial y_1} & & & \\ & & \ddots & & \\ & & & \frac{\partial f_{\emptyset_m}}{\partial x_m} & \frac{\partial f_{\emptyset_m}}{\partial y_m} \\ & & & \frac{\partial \overline{f}_{\emptyset_m}}{\partial x_m} & \frac{\partial \overline{f}_{\emptyset_m}}{\partial y_m} \end{pmatrix} =$$

$$det\begin{pmatrix} \frac{\partial}{\partial x_1}(x_1+iy_1) & \frac{\partial}{\partial y_1}(x_1+iy_1) & & & \\ \frac{\partial}{\partial x_1}(x_1-iy_1) & \frac{\partial}{\partial y_1}(x_1-iy_1) & & & \\ & & \ddots & & \\ & & & \frac{\partial}{\partial x_m}(x_m+iy_m) & \frac{\partial}{\partial y_m}(x_m+iy_m) \\ & & & \frac{\partial}{\partial x_m}(x_m-iy_m) & \frac{\partial}{\partial y_m}(x_m-iy_m) \end{pmatrix} =$$

$$det\begin{pmatrix} 1 & i & & & \\ 1 & -i & & & \\ & & \ddots & & \\ & & & 1 & i \\ & & & 1 & -i \end{pmatrix} = (-2i)^m ,$$

then we have that the equation (3.10) is equal to:

$$\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{det\,(JV_\emptyset)}(p_j)\int_{\mathbb{R}^{2m}}\exp\left\{-x_1^2-y_1^2-\cdots-x_m^2-y_m^2\right\}\cdot(-2i)^m dx_1\wedge dy_1\wedge\cdots\wedge dx_m\wedge dy_m =$$
$$(-2i)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{det\,(JV_\emptyset)}(p_j)\times$$
$$\int_{-\infty}^{\infty}\exp\left\{-x_1^2\right\}dx_1\int_{-\infty}^{\infty}\exp\left\{-y_1^2\right\}dy_1\cdots\int_{-\infty}^{\infty}\exp\left\{-x_m^2\right\}dx_m\int_{-\infty}^{\infty}\exp\left\{-y_m^2\right\}dy_m =$$
$$(-2i)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{det\,(JV_\emptyset)}(p_j)\left(\sqrt{\pi}\right)^{2m} = (-2\pi i)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{det\,(JV_\emptyset)}(p_j) = \left(\frac{2\pi}{i}\right)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{det\,(JV_\emptyset)}(p_j)\,.$$

Then:

$$\lim_{t\to 0}\int_{SB_\epsilon(p_j)}\exp\left\{-\frac{\overline{\partial}\omega_\emptyset}{t}-\frac{i_{V_\emptyset}(\omega_\emptyset)}{t}\right\}\cdot\eta = \left(\frac{2\pi}{i}\right)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\dots,n;1,\dots,n)}}{\det(JV_\emptyset)}(p_j)\,.$$

Since $p_j \in Sing(V)$ is non-degenerate, by lemma 3.18 and by remark 3.19, we conclude that:

$$\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\det(JV_{\emptyset})}(p_j) \simeq \left[\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j)\,.$$

Therefore:

$$\lim_{t\to 0}\int_{SB_{\epsilon}(p_j)}\exp\left\{-\frac{\overline{\partial}\omega_{\emptyset}}{t}-\frac{i_{V_{\emptyset}}(\omega_{\emptyset})}{t}\right\}\cdot\eta = \left(\frac{2\pi}{i}\right)^m\left[\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j).$$

Then:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^m\left[\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j).$$

□

Corollary 3.21. Let V be a even holomorphic vector field with a non-degenerate isolated singularity $p_j \in Sing(V)$, let S be a compact complex supermanifold with $m|n$ dimension and let $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial}+i_V)(\eta)=0$. If $\det(\mathbb{B}(D))(\mathbb{B}(p_j))=1$, then we have:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^m\frac{\eta^{(0,0)|(n,n)}_{(1,\ldots,n;1,\ldots,n)}}{\mathrm{Ber}(V)}(p_j)$$

3.4 Localization formula for odd vectors

We will begin this section with some definitions and technical lemmas.

Lemma 3.22. If dO_1 and dO_2 are both odd 1-forms, and if dE_1 and dE_2 are both even 1-forms, then the wedge product between 2-forms composed by 1-forms with inverse parities are anticommutatives, i.e.,

$$\begin{aligned}
(dE_i\wedge dO_j)\wedge(dE_k\wedge dO_l) &= -(dE_k\wedge dO_l)\wedge(dE_i\wedge dO_j)\\
(dO_j\wedge dE_i)\wedge(dE_k\wedge dO_l) &= -(dE_k\wedge dO_l)\wedge(dO_j\wedge dE_i)\\
(dE_i\wedge dO_j)\wedge(dO_l\wedge dE_k) &= -(dO_l\wedge dE_k)\wedge(dE_i\wedge dO_j)\\
(dO_j\wedge dE_i)\wedge(dO_l\wedge dE_k) &= -(dO_l\wedge dE_k)\wedge(dO_j\wedge dE_i)
\end{aligned}$$

with $i,k\in\{1,2\}, i\neq k$ and $j,l\in\{1,2\}$.

Proof. This is a direct consequence of the following facts (see proposition 1.75):

$$\begin{aligned} dE_i \wedge dE_k &= -(-1)^{\langle \varepsilon(dE_i)|\varepsilon(dE_k)\rangle} dE_k \wedge dE_i &= -dE_k \wedge dE_i \\ dE_i \wedge dO_j &= -(-1)^{\langle \varepsilon(dE_i)|\varepsilon(dO_j)\rangle} dO_j \wedge dE_i &= -dO_j \wedge dE_i \\ dO_j \wedge dO_l &= -(-1)^{\langle \varepsilon(dO_j)|\varepsilon(dO_l)\rangle} dO_l \wedge dO_j &= dO_l \wedge dO_j. \end{aligned}$$

□

Definition 3.23. Given, in local coordinates, $\omega = \sum_{i=1}^n dz_i \overline{g}_i + \sum_{i=1}^n d\xi_i \overline{g}_i$, then we define ω_1 and ω_2 as:

$$\omega_1 := \sum_{i=1}^n dz_i \overline{g}_i \quad \textit{and} \quad \omega_2 := \sum_{i=1}^n d\xi_i \overline{g}_i$$

Lemma 3.24. Let $V = \sum_{i=1}^n f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i \frac{\partial}{\partial \xi_i}$ be a holomorphic odd vector written in local coordinates, where the f_i are holomorphic odd functions and g_i are holomorphic even functions without odd variables. Then, given $\omega = \sum_{i=1}^n dz_i \overline{g}_i + \sum_{i=1}^n d\xi_i \overline{g}_i$, we have:

(i) $\overline{\partial}\omega = \sum_{i=1}^n d\overline{g}_i \wedge dz_i + \sum_{i=1}^n d\overline{g}_i \wedge d\xi_i$;

(ii) $\overline{\partial}\omega_1 = \sum_{i=1}^n d\overline{g}_i \wedge dz_i$ and $\overline{\partial}\omega_2 = \sum_{i=1}^n d\overline{g}_i \wedge d\xi_i$;

(iii) $(\overline{\partial}\omega_1)^{n-1} = \sum_{j=1}^n (n-1)!\ d\overline{g}_1 \wedge dz_1 \wedge \cdots \wedge \widehat{d\overline{g}_j \wedge dz_j} \wedge \cdots \wedge d\overline{g}_n \wedge dz_n$;

(iv) $(\overline{\partial}\omega_1)^n = (-1)^{\frac{n(n+1)}{2}} \frac{n!}{det(JV)}\ dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n$, where $JV = \left(\frac{\partial g_i}{\partial z_j}\right)_{n\times n}$;

(v) $(\overline{\partial}\omega_2)^2 = 0$;

(vi) $(\overline{\partial}\omega_1)^{n-1} \cdot\ \overline{\partial}\omega_2 = \sum_{j=1}^n (n-1)!\ (-1)^{\frac{n(n+1)}{2}}\ dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_j \wedge \cdots \wedge d\overline{g}_n$.

Proof. (i) $\omega = \sum_{i=1}^n dz_i \overline{g}_i + \sum_{i=1}^n d\xi_i \overline{g}_i$, then, by equation (2.3), we have:

$$\overline{\partial}\omega = \sum_{i=1}^n -dz_i \wedge d\overline{g}_i + \sum_{i=1}^n -d\xi_i \wedge d\overline{g}_i = \sum_{i=1}^n d\overline{g}_i \wedge dz_i + \sum_{i=1}^n d\overline{g}_i \wedge d\xi_i$$

(ii) It's a direct consequence of (i).

(iii) $(\overline{\partial}\omega_1)^{n-1} = (\sum d\overline{g}_i \wedge dz_i)^{n-1} = \sum_{j=1}^n (n-1)!\ d\overline{g}_1 \wedge dz_1 \wedge \cdots \wedge \widehat{d\overline{g}_j \wedge dz_j} \wedge \cdots \wedge d\overline{g}_n \wedge dz_n$.

(iv)

$$(\overline{\partial}\omega_1)^n = \left(\sum d\overline{g}_i \wedge dz_i\right)^n = (n)!\ d\overline{g}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{g}_j \wedge dz_j \wedge \cdots \wedge d\overline{g}_n \wedge dz_n$$
$$(\overline{\partial}\omega_1)^n = (n)!(-1)^{\frac{n(n+1)}{2}}\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n$$
$$(\overline{\partial}\omega_1)^n = (-1)^{\frac{n(n+1)}{2}} \frac{(n)!}{det(JV)}\ dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n$$

where $JV = \left(\frac{\partial g_i}{\partial z_j}\right)_{n\times n}$.

(v) As a direct consequence of lemma 3.22, we have $\left(\sum d\overline{g}_i \wedge d\xi_i\right)^2 = 0$. Thus

$$(\overline{\partial}\omega_2)^2 = \left(\sum d\overline{g}_i \wedge d\xi_i\right)^2 = 0.$$

(vi)

$$(\overline{\partial}\omega_1)^{n-1} \cdot\ \overline{\partial}\omega_2 =$$
$$\left(\sum_{j=1}^{n}(n-1)!\ d\overline{g}_1 \wedge dz_1 \wedge \cdots \wedge \widehat{d\overline{g}_j \wedge dz_j} \wedge \cdots \wedge d\overline{g}_n \wedge dz_n\right) \cdot \left(\sum_{i=1}^{n} d\overline{g}_i \wedge d\xi_i\right).$$
$$(\overline{\partial}\omega_1)^{n-1} \cdot\ \overline{\partial}\omega_2 = \left(\sum_{j=1}^{n}(n-1)!\ d\overline{g}_1 \wedge dz_1 \wedge \cdots \wedge d\overline{g}_j \wedge d\xi_j \wedge \cdots \wedge d\overline{g}_n \wedge dz_n\right)$$
$$(\overline{\partial}\omega_1)^{n-1} \cdot \overline{\partial}\omega_2 = \sum_{j=1}^{n}(n-1)!\,(-1)^{\frac{n(n+1)}{2}}\ dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_j \wedge \cdots \wedge d\overline{g}_n.$$

□

Remark 3.25. Since the functions $g_i, i \in \{1, \ldots, n\}$, are even functions, then, in this case, it's no problem to represent ω, ω_1 and ω_2 by

$$\omega = \sum_{i=1}^{n} \overline{g}_i dz_i + \sum_{i=1}^{n} \overline{g}_i d\xi_i, \quad \omega_1 = \sum_{i=1}^{n} \overline{g}_i dz_i, \quad \textit{and} \quad \omega_2 = \sum_{i=1}^{n} \overline{g}_i d\xi_i.$$

Lemma 3.26. Let $V = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial \xi_i}$ be an holomorphic odd vector written in local coordinates, where f is an holomorphic odd function and g an holomorphic even function without odd variables. Then, given $\omega = \sum_{i=1}^{n} \overline{g}_i dz_i + \sum_{i=1}^{n} \overline{g}_i d\xi_i$, we have:

(i) $\exp\{-i_V(\omega)\} = \exp\{-\sum \overline{g}_i g_i\} \cdot (1 - \sum \overline{g}_i f_i)$;

(ii) $\exp\{\sum_{i=1}^{n} d\overline{g}_i \wedge dz_i + \sum_{i=1}^{n} d\overline{g}_i \wedge d\xi_i\} = \exp\{\sum_{i=1}^{n} d\overline{g}_i \wedge dz_i\} \cdot (1 + \sum_{i=1}^{n} d\overline{g}_i \wedge d\xi_i)$;

(iii) $\exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} = \exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\} + \sum_{j=1}^{n} \frac{(-1)^j}{(j-1)!} \frac{\left(\overline{\partial}\omega_1\right)^{(j-1)}\ \overline{\partial}\omega_2}{t^j}$.

Proof. (i)

$$\exp\{-i_V(\omega)\} = \exp\left\{-\sum \overline{g}_i g_i - \sum \overline{g}_i f_i\right\} = \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \exp\left\{-\sum \overline{g}_i f_i\right\} =$$
$$\exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \left(1 - \sum \overline{g}_i f_i + \frac{(\sum \overline{g}_i f_i)^2}{2!} - \frac{(\sum \overline{g}_i f_i)^3}{3!} + \cdots\right)$$

But observe that the function $\sum \overline{g}_i f_i$ is odd, and thus is nilpotent, i.e:

$$\left(\sum \overline{g}_i f_i\right)^2 = \left(\sum \overline{g}_i f_i\right)^3 = \cdots = \left(\sum \overline{g}_i f_i\right)^k = \cdots = 0.$$

So, the result follows.

(ii)

$$\exp\left\{\sum_{i=1}^n d\overline{g}_i \wedge dz_i + \sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right\} = \exp\left\{\sum_{i=1}^n d\overline{g}_i \wedge dz_i\right\} \cdot \exp\left\{\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right\} =$$
$$\exp\left\{\sum_{i=1}^n d\overline{g}_i \wedge dz_i\right\} \cdot \left(1 + \left(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right) + \frac{(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i)^2}{2!} + \frac{(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i)^3}{3!} + \cdots\right)$$

However, as direct consequence of the lemma 3.22, we have:

$$\left(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right)^2 = \left(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right)^3 = \cdots = \left(\sum_{i=1}^n d\overline{g}_i \wedge d\xi_i\right)^k = \cdots = 0.$$

So, the result follows.

(iii) As consequence of the item (ii), we have:

$$\exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} = \exp\left\{-\frac{\overline{\partial}\omega_1}{t} - \frac{\overline{\partial}\omega_2}{t}\right\} = \exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\} \cdot \exp\left\{-\frac{\overline{\partial}\omega_2}{t}\right\} =$$
$$\left\{1 + \cdots + \frac{1}{(k-1)!}\left(-\frac{\overline{\partial}\omega_1}{t}\right)^{k-1} + \frac{1}{k!}\left(-\frac{\overline{\partial}\omega_1}{t}\right)^k + \cdots\right\}\left\{1 - \frac{\overline{\partial}\omega_2}{t}\right\} = \exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\} +$$
$$\left\{-\frac{\overline{\partial}\omega_2}{t} + \frac{\overline{\partial}\omega_1 \cdot \overline{\partial}\omega_2}{t^2} - \frac{1}{2!}\frac{\left(\overline{\partial}\omega_1\right)^2 \cdot \overline{\partial}\omega_2}{t^3} + \cdots \right.$$
$$\left. \cdots + \frac{(-1)^n}{(n-1)!}\frac{\left(\overline{\partial}\omega_1\right)^{n-1} \cdot \overline{\partial}\omega_2}{t^n} + \frac{(-1)^{n+1}}{(n)!}\frac{\left(\overline{\partial}\omega_1\right)^n \cdot \overline{\partial}\omega_2}{t^{n+1}}\right\}.$$

And since $\frac{(-1)^{n+1}}{(n)!}\frac{\left(\overline{\partial}\omega_1\right)^n\cdot\overline{\partial}\omega_2}{t^{n+1}}=0$, we find:

$$\exp\left\{-\frac{\overline{\partial}\omega}{t}\right\}=\exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\}+\sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!}\frac{\left(\overline{\partial}\omega_1\right)^{(j-1)}\ \overline{\partial}\omega_2}{t^j}$$

□

Definition 3.27. Let us to define some notations that will be important to the next lemma and theorems:

1. We define the function $\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$ as the function that go along with the form:

$$dz_j\wedge d\xi_1\wedge\cdots\wedge\widehat{d\xi_j}\wedge\cdots\wedge d\xi_n\wedge d\overline{\xi}_1\wedge\cdots\wedge d\overline{\xi}_n\ \eta_{\widehat{j}}^{(1,0)|(n-1,n)}$$

2. We define the function $\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ as the sum of the functions $\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$, i.e.:

$$\eta_{\widehat{*}}^{(1,0)|(n-1,n)}=\sum_{j=1}^{n}\ \eta_{\widehat{j}}^{(1,0)|(n-1,n)}$$

3. We define ${}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ as the even function given by:

$${}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}=\sum_{j=1}^{n}\ {}^{E}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$$

   where each parcel ${}^{E}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$ is an even function.

4. We define ${}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ as the odd function given by:

$${}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}=\sum_{j=1}^{n}\ {}^{O}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$$

   where each parcel ${}^{O}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$ is an odd function.

And as consequence of the items (3) and (4), to a non homogeneous function $\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$, we have:

$$\eta_{\widehat{*}}^{(1,0)|(n-1,n)}=\ {}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}\oplus\ {}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}\tag{3.11}$$

Remark 3.28. Let us to comment some important details to comprehension the proofs of the next theorems:

- As ${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ is an odd function, then the term $\left(\xi_1 \cdots \xi_n \cdot \overline{\xi}_1 \cdots \overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*(1\ldots n,1\ldots n)}}\right)$ doesn't belong to its expansion because this term is even;
- If $L(\mu)$ is an even number (μ is a multi-index) and n is an odd number, or if $L(\mu)$ is an odd number (μ is a multi-index) and n is an even number, then the term $\left(\xi^{\mu} \cdot \overline{\xi}^{1} \cdots \overline{\xi}^{n} \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*(\mu,1\ldots n)}}\right)$ **does not belong** to the expansion of ${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ because this term is odd, but the function ${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ is even.

Lemma 3.29. With the notations of the definition 3.27, we have:

$$\sum_{j=1}^{n} dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; \eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge$$
$$\wedge\, dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$$
$$-\; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}.$$

Proof. Remembering that $\eta^{(1,0)|(n-1,n)}_{\widehat{j}} = {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} + {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}}$, and using proposition 1.75, let us to begin studying the following factor:

$$dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; \eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; \left({}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} + {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}}\right) \wedge dz_1 \wedge$$
$$\cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n +$$
$$dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$-d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} dz_j \wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n$$
$$-d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} dz_j \wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} d\xi_j \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n +$$
$$d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} d\xi_j \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$
$$d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n$$
$$-\, d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{j}} \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n =$$

$$d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \; {}^{E}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}$$
$$- d\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \wedge dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \; {}^{O}\eta_{\widehat{j}}^{(1,0)|(n-1,n)}.$$

Then:

$$\sum_{j=1}^{n} dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \; \eta_{\widehat{j}}^{(1,0)|(n-1,n)} \wedge$$
$$\wedge\, dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n =$$
$$d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \left(\sum_{j=1}^{n} {}^{E}\eta_{\widehat{j}}^{(1,0)|(n-1,n)} \right)$$
$$-\; d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \left(\sum_{j=1}^{n} {}^{O}\eta_{\widehat{j}}^{(1,0)|(n-1,n)} \right).$$

And by definition 3.27, items (3) and (4), we have:

$$\sum_{j=1}^{n} dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \; \eta_{\widehat{j}}^{(1,0)|(n-1,n)} \wedge$$
$$\wedge\, dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n =$$
$$dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \; {}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$$
$$-\; dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \; {}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}.$$

□

Now, consider the odd vector field in the superball $SB_\epsilon(p_i)$:

$$V = \sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} g_j \frac{\partial}{\partial \xi_j} \tag{3.12}$$

with $f_j(z,\xi)$ holomorphic odd functions and $g_j(z)$ holomorphic even functions without odd variables, i.e, even functions with expansion $g_j(z,\xi) = \widehat{g}_{j_\emptyset}(z) + \sum_{\lambda \in M} \xi^\lambda \cdot 0$.

Definition 3.30. Let S be a supermanifold of $n|n$ dimension and let V be an odd supervector on S written locally as done in the equation (3.12). Then, we define:

$$Sing(V) := \{p \in S \mid g_i(\mathbb{B}(p)) = 0,\ \forall i \in \{1, \dots, n\}\}$$

$Sing(V)$ is called singular set of V on S and $\forall p \in Sing(V)$ is called singular point or singularity of V on S.

Definition 3.31. We say that $Sing(V)$ is a set of isolated singularities if $\forall p_i \in Sing(V)$ $\exists\ \epsilon > 0$ such that $B_\epsilon\left(\mathbb{B}(p_i)\right) \cap \mathbb{B}\left(Sing(V)\right) = \mathbb{B}(p_i)$.

Definition 3.32. Let S be a supermanifold of $n|n$ dimension, let V be an odd supervector on S, written in local coordinates by $V = \sum_{i=1}^n f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i \frac{\partial}{\partial \xi_i}$, with f_i odd functions e g_i even functions. Then we define the Berezinian of the odd vector V by

$$\mathrm{Ber}(V)(z,\xi) = \mathrm{sdet}\begin{pmatrix} \frac{\partial g_i}{\partial z_j} & \frac{\partial f_k}{\partial z_j} \\ \frac{\partial g_i}{\partial \xi_l} & \frac{\partial f_k}{\partial \xi_l} \end{pmatrix}$$

Definition 3.33. Let V be an odd supervector on S. We say that $p \in Sing(V)$ is a non-degenerate singularity of V if $\mathbb{B}\left[\mathrm{Ber}(V)\right](\mathbb{B}(p))$ exists and $\mathbb{B}\left[\mathrm{Ber}(V)\right](\mathbb{B}(p)) \neq 0$.

Let V be an odd supervector with isolated singularities. Then, for all $p_\kappa \in Sing(V)$, choose superballs $SB_\epsilon(p_\kappa)$ (with $B_\epsilon(\mathbb{B}(p_\kappa)) = \mathbb{B}(SB_\epsilon(p_\kappa))$) such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) = \emptyset$, for each $\mathbb{B}(p_i) \neq \mathbb{B}(p_j)$, with $p_i, p_j \in Sing(V)$.

Remark 3.34. If $\mathbb{B}(p_i) = \mathbb{B}(p_j)$, then $SB_\epsilon(p_i) = SB_\epsilon(p_j)$ because we are working in the DeWitt topology (see definition 2.7), and, therefore, any superball cover all the odd part of the supermanifold.

Now, assuming the conditions over the superballs, consider the form defined in $SB_\epsilon(p_i)$:

$$\widetilde{\omega}_i = \sum_{j=1}^n \overline{g}_j dz_j + \sum_{j=1}^n \overline{g}_j d\xi_j \tag{3.13}$$

Observe that each $\widetilde{\omega}_i$ is defined only locally, because the functions g_i are holomorphic on the compact supermanifold S. If they were defined globally, then they should be constant functions ([19], [12]). We will use the unit partition to define a global form ω on S.

With an appropriate partition of unity, we will construct a global form ω such that

$$\omega_{|_{SB_\epsilon(p_i)}} = \widetilde{\omega}_i,\ \forall i.$$

Definition 3.35. Let S be a compact supermanifold of $n|n$ dimension, and let $SB_\epsilon(p_i)$ be superballs in S. Now, let $\mathfrak{U} = \left(S - \cup_i \overline{SB}_\epsilon(p_i)\right) \bigcup (\cup_i SB_{2\epsilon}(p_i))$ be an open cover of S, such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) = \emptyset$, for $p_i, p_j \in Sing(V)$, let ω^0 be a form defined on $\left(S - \cup_i \overline{SB}_\epsilon(p_i)\right)$ such that $\mathbb{B}\left[i_V(\omega^0)\right] > 0$ and let $\{\rho_i\}$ be a partition of unity subordinated to $\mathfrak{U}$ (see theorem 2.13). Then we define ω as the global form given by:

$$\omega = \rho_0 \omega^0 + \sum_i \rho_i \widetilde{\omega}_i. \tag{3.14}$$

To complete the definition, we need to construct the form ω^0. Let $\mathfrak{V} = \bigcup_j W_j$ be an open cover of $\overline{S - \cup_i \overline{SB}_\epsilon(p_i)}$, where, for each W_j, we have $V = \sum_{i=1}^n f_i^j \frac{\partial}{\partial z_i} + \sum_{i=1}^n g_i^j \frac{\partial}{\partial \xi_i}$, and let $\{\mu\}$ be a partition of unity subordinated to $\mathfrak{V}$. Then, we construct ω^0 as follows:

$$\omega^0 = \sum_j \mu_j \left(\sum_{i=1}^n \overline{g}_i^j d\xi_i \right). \tag{3.15}$$

With this construction, we ensure that $\mathbb{B}\left[i_V(\omega^0)\right] > 0 \quad \forall p \in S - \cup_i \overline{SB}_\epsilon(p_i)$.

Remark 3.36. The operation $S - \cup_i \overline{SB}_\epsilon(p_i)$ is defined as follows:

$$S - \cup_i \overline{SB}_\epsilon(p_i) := \mathbb{B}^{-1}\left(X - \cup_i \overline{B}_\epsilon\left(\mathbb{B}(p_i)\right)\right)$$

where $X = \mathbb{B}(S)$.

Lemma 3.37.

$$(\overline{\partial} + i_V)^2(\widetilde{\omega}_i) = 0, \ \forall i \quad \textit{and} \quad (\overline{\partial} + i_V)^2(\omega^0) = 0.$$

Proof. Consider $\widetilde{\omega}_i = \sum_{i=j}^n \overline{g}_j dz_j + \sum_{j=1}^n \overline{g}_j d\xi_j$. Then, by the fact that f, g are holomorphic functions together with the properties of the contraction operator (see [13]) and of the derivation, it follows that:

(In this proof, we will use the proper: $i_V(\alpha \wedge \beta) = i_V(\alpha) \wedge \beta + (-1)^\kappa \alpha \wedge i_V(\beta)$, with α being a κ-form)

$$\begin{aligned}
(\overline{\partial}+i_V)^2(\widetilde{\omega}_i) &= (\overline{\partial}+i_V)^2 \left(\sum \overline{g}_j dz_j + \sum \overline{g}_j d\xi_j\right) = (\overline{\partial}+i_V)(\overline{\partial}+i_V)\left(\sum \overline{g}_j dz_j + \sum \overline{g}_j d\xi_j\right) = \\
&(\overline{\partial}+i_V)(\overline{\partial})\left(\sum \overline{g}_j dz_j + \sum \overline{g}_j d\xi_j\right) + (\overline{\partial}+i_V)(i_V)\left(\sum \overline{g}_j dz_j + \sum \overline{g}_j d\xi_j\right) = \\
&(\overline{\partial}+i_V)\left(\sum d\overline{g}_j \wedge dz_j + \sum d\overline{g}_j \wedge d\xi_j + \sum \overline{g}_j f_j + \sum \overline{g}_j g_j\right) = \\
&(\overline{\partial})\left(\sum d\overline{g}_j \wedge dz_j + \sum d\overline{g}_j \wedge d\xi_j + \sum \overline{g}_j f_j + \sum \overline{g}_j g_j\right) + \\
&(i_V)\left(\sum d\overline{g}_j \wedge dz_j + \sum d\overline{g}_j \wedge d\xi_j + \sum \overline{g}_j f_j + \sum \overline{g}_j g_j\right) =
\end{aligned}$$

$$\sum\left[\overline{\partial}\overline{g}_j f_j+\overline{g}_j\overline{\partial}f_j\right]+\sum\left[\overline{\partial}\overline{g}_j g_j+\overline{g}_j\overline{\partial}g_j\right]+$$
$$\sum\left(i_V(d\overline{g}_j)\wedge dz_j-d\overline{g}_j\wedge i_V(dz_j)\right)+\sum\left(i_V(d\overline{g}_j)\wedge d\xi_j-d\overline{g}_j\wedge i_V(d\xi_j)\right)+$$
$$\sum i_V\left(\overline{g}_j f_j\right)+\sum i_V\left(\overline{g}_j g_j\right) \quad (3.16)$$

Since $\overline{\partial}f_j=\overline{\partial}g_j=0$ (because f_j, g_j are holomorphic functions), $i_V(d\overline{g}_j)=0$, and, by convencional (see [13]), $i_V(\overline{g}_j f_j)=i_V(\overline{g}_j g_j)=0$, we have that the equation 3.16 is equal to:

$$=\left(\sum f_j d\overline{g}_j+\sum g_j d\overline{g}_j\right)+\left(-\sum f_j d\overline{g}_j-\sum g_j d\overline{g}_j\right)=0.$$

Now, let us to prove that $(\overline{\partial}+i_V)^2(\omega^0)=0$.

$$(\overline{\partial}+i_V)^2\left(\sum_j\mu_j\left(\sum_{i=1}^n\overline{g}_i^j d\xi_i\right)\right)=\sum_j\mu_j\left(\sum_{i=1}^n(\overline{\partial}+i_V)^2(\overline{g}_i^j d\xi_i)\right)$$

And, for each term, we have:

$$(\overline{\partial}+i_V)\left(d\overline{g}_i^j\wedge d\xi_i+\overline{g}_i^j g_i^j\right)=\overline{\partial}\left(d\overline{g}_i^j\wedge d\xi_i+\overline{g}_i^j g_i^j\right)+i_V\left(d\overline{g}_i^j\wedge d\xi_i+\overline{g}_i^j g_i^j\right)=$$
$$\overline{\partial}\overline{g}_i^j g_i^j+\overline{g}_i^j\overline{\partial}g_i^j+i_V(d\overline{g}_i^j)\wedge d\xi_i-d\overline{g}_i^j\wedge i_V(d\xi_i)=g_i^j d\overline{g}_i^j-g_i^j d\overline{g}_i^j=0.$$

□

Remark 3.38. By equation (3.13), lemma 3.37 and definiton 3.35, we observe that $\omega_{|_{SB_\epsilon(p_i)}}=\widetilde{\omega}_i,\ \forall i$, and $(\overline{\partial}+i_V)^2(\omega)=0$.

From now on, the odd holomorphic vector V and the form ω will be given by the equation (3.12) and by the definition 3.35, respectively. And remember that $g(z)$ in the equation (3.12) is a function without odd variables.

Lemma 3.39. Let S be a supermanifold of $n|n$ dimension. If $t>0$ and if $\eta\in\bigoplus A^{(p,q)|(r,s)}$ is a form such that $(\overline{\partial}+i_V)(\eta)=0$, then we have:

$$\int_S\eta=\int_S\eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t}-\frac{i_V(\omega)}{t}\right\}$$

Proof.

$$\frac{\partial}{\partial\kappa}\int_S\eta\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}=-\int_S\eta\cdot(\overline{\partial}+i_V)\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}.$$

Now, observe that, by remark 3.38 $\left((\overline{\partial}+i_V)^2(\omega)=0\right)$, we have:

$$(\overline{\partial}+i_V)\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}=\sum_L(\overline{\partial}+i_V)\frac{\left(-\kappa(\overline{\partial}+i_V)\omega\right)^L}{L!}=0$$

Therefore, as $(\overline{\partial}+i_V)\eta=0$ and $(\overline{\partial}+i_V)\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}=0$, we have:

$$-\int_S\eta\cdot(\overline{\partial}+i_V)\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}=-\int_S(\overline{\partial}+i_V)\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right).$$

Then:

$$\int_S(\overline{\partial}+i_V)\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right)=\int_S\overline{\partial}\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right)+\int_S i_V\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right).$$

By lemma 3.6, we have $\int_S\overline{\partial}\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right)=0$ and as the contraction i_V promotes the lost of top form in the integral, then $\int_S i_V\left(\eta\cdot\omega\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}\right)=0$.

Therefore $\frac{\partial}{\partial\kappa}\int_S\eta\cdot\exp\left\{-\kappa(\overline{\partial}+i_V)\omega\right\}=0$. So, we have:

$$\int_S\eta=\int_S\eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t}-\frac{i_V(\omega)}{t}\right\}$$

□

Theorem 3.40. If the odd holomorphic vector field V is a supervector without singularities on a supermanifold S of $n|n$ dimension, then for any $\eta\in\oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial}+i_V)(\eta)=0$, we have:

$$\int_S\eta=0.$$

Proof. By definition 3.35 , we have:

$$i_V(\omega)=\left[\rho_o\omega^0+\sum_i\rho_i\left(\sum_j\overline{g}^i_j d\xi_j\right)\right](V)+\left[\sum_i\rho_i\left(\sum_j\overline{g}^i_j dz_j\right)\right](V)\tag{3.17}$$

To simplify the notation, we will use the following:

$$\omega^v=\left(\rho_o\omega^0+\sum_i\rho_i\sum_j\overline{g}^i_j d\xi_j\right)(V).\tag{3.18}$$

Since V is a supervector without singularities on S, then $\mathbb{B}\left(\omega^v\right)>0$, and as S is compact, $\exists \delta > 0$ such that:

$$\mathbb{B}\left(\omega^{v}\right) > \delta. \tag{3.19}$$

Now, note that:

$$\left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j dz_j\right)\right](V) = \left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j f^i_j\right)\right]. \tag{3.20}$$

Beyond that, we have:

$$\overline{\partial}\omega = \rho_0 \sum_k \mu_k \left(\sum_{j=1}^{n} d\overline{g}^k_j \wedge d\xi_j\right) + \sum_i \rho_i \left(\sum_j^{n} d\overline{g}^i_j \wedge d\xi_j\right) + \sum_i \rho_i \left(\sum_{j=1}^{n} d\overline{g}^i_j \wedge dz_j\right) \tag{3.21}$$

Then, by lemma 3.39:

$$\left|\int_S \eta\right| = \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\}\right| = \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} \exp\left\{\frac{-i_V(\omega)}{t}\right\}\right|.$$

By equation (3.17), we have:

$$= \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} \exp\left\{-\frac{\left[\rho_o\omega^0 + \sum_i \rho_i \left(\sum_j \overline{g}^i_j d\xi_j\right)\right](V)}{t} - \frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j dz_j\right)\right](V)}{t}\right\}\right|.$$

By notation (3.18), we have:

$$= \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} \exp\left\{-\frac{\omega^v}{t} - \frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j dz_j\right)\right](V)}{t}\right\}\right|$$

Let's denote $\exp\left\{-\frac{\omega^v}{t}\right\}$ by $e^{-\omega^v/t}$. Then:

$$= \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} \exp\left\{-\frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j dz_j\right)\right](V)}{t}\right\} e^{-\omega^v/t}\right|.$$

By equation (3.20):

$$= \left|\int_S \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t}\right\} \exp\left\{-\frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}^i_j f^i_j\right)\right]}{t}\right\} e^{-\omega^v/t}\right|.$$

By the demonstration of lemma 3.26, item (i), we have:

$$= \left| \int_S \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} \right\} \left(1 - \frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}_j^i f_j^i \right) \right]}{t} \right) e^{-\omega^v/t} \right| \leq$$

$$\left| \int_S \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} \right\} e^{-\omega^v/t} \right| + \left| \int_S \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} \right\} \cdot \frac{\left[\sum_i \rho_i \left(\sum_j \overline{g}_j^i f_j^i \right) \right]}{t} e^{-\omega^v/t} \right| \leq$$

$$\left| \int_S \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} \right\} e^{-\omega^v/t} \right| + \sum_a \sum_b \left| \int_S \eta \cdot \exp\left\{ -\frac{\overline{\partial}\omega}{t} \right\} \cdot \frac{\left[\rho_a \left(\overline{g}_b^a f_b^a \right) \right]}{t} e^{-\omega^v/t} \right|.$$

By equation (3.21):

$$= \left| \int_S \eta \cdot \exp\left\{ \frac{-\rho_0 \sum_k \mu_k \left(\sum_{j=1}^n d\overline{g}_j^k \wedge d\xi_j \right) - \sum_i \rho_i \left(\sum_j^n d\overline{g}_j^i \wedge d\xi_j \right) - \sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right\} e^{-\omega^v/t} \right|$$

$$+ \sum_a \sum_b \left| \int_S \eta \cdot \exp\left\{ \frac{-\rho_0 \sum_k \mu_k \left(\sum_{j=1}^n d\overline{g}_j^k \wedge d\xi_j \right) - \sum_i \rho_i \left(\sum_j^n d\overline{g}_j^i \wedge d\xi_j \right) - \sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right\} \times \frac{\left[\rho_a \left(\overline{g}_b^a f_b^a \right) \right]}{t} e^{-\omega^v/t} \right|$$

By lemma 3.26, item (ii):

$$= \left| \int_S \eta \cdot \exp\left\{ \frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right\} \times \left(1 - \frac{\rho_0 \sum_k \mu_k \left(\sum_{j=1}^n d\overline{g}_j^k \wedge d\xi_j \right)}{t} - \frac{\sum_i \rho_i \left(\sum_j^n d\overline{g}_j^i \wedge d\xi_j \right)}{t} \right) e^{-\omega^v/t} \right| +$$

$$\sum_a \sum_b \left| \int_S \eta \cdot \exp\left\{ \frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right\} \times \left(1 - \frac{\rho_0 \sum_k \mu_k \left(\sum_{j=1}^n d\overline{g}_j^k \wedge d\xi_j \right)}{t} - \frac{\sum_i \rho_i \left(\sum_j^n d\overline{g}_j^i \wedge d\xi_j \right)}{t} \right) \cdot \frac{\left[\rho_a \left(\overline{g}_b^a f_b^a \right) \right]}{t} e^{-\omega^v/t} \right| \leq$$

$$\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}e^{-\omega^v/t}\right|+$$

$$\sum_k\sum_j\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}\frac{\rho_0\mu_k\left(d\overline{g}_j^k\wedge d\xi_j\right)}{t}e^{-\omega^v/t}\right|+$$

$$\sum_i\sum_j\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}\frac{\rho_i\left(d\overline{g}_j^i\wedge d\xi_j\right)}{t}e^{-\omega^v/t}\right|+$$

$$\sum_a\sum_b\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}\frac{[\rho_a\left(\overline{g}_b^a f_b^a\right)]}{t}e^{-\omega^v/t}\right|+$$

$$\sum_a\sum_b\sum_k\sum_l\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}\frac{\rho_0\mu_k\left(d\overline{g}_l^k\wedge d\xi_l\right)}{t}\frac{[\rho_a\left(\overline{g}_b^a f_b^a\right)]}{t}e^{-\omega^v/t}\right|+$$

$$\sum_a\sum_b\sum_k\sum_l\left|\int_S \eta\cdot\exp\left\{\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right\}\frac{\rho_k\left(d\overline{g}_l^k\wedge d\xi_l\right)}{t}\frac{[\rho_a\left(\overline{g}_b^a f_b^a\right)]}{t}e^{-\omega^v/t}\right|\leq$$

$$\sum_{\alpha=0}^n\frac{1}{\alpha!}\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\cdot\eta_{(1\dots n,1\dots n)}^{(n-\alpha,n-\alpha)|(n,n)}e^{-\omega^v/t}\right|+$$

$$\sum_{\alpha=0}^{n-1}\frac{1}{\alpha!}\sum_k\sum_j\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\frac{\rho_0\mu_k d\overline{g}_j^k}{t}\cdot\eta_{(1\dots n,1\dots n)}^{(n-\alpha,n-\alpha-1)|(n-1,n)}e^{-\omega^v/t}\right|+$$

$$\sum_{\alpha=0}^{n-1}\frac{1}{\alpha!}\sum_i\sum_j\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\frac{\rho_i d\overline{g}_j^i}{t}\cdot\eta_{(1\dots n,1\dots n)}^{(n-\alpha,n-\alpha-1)|(n-1,n)}e^{-\omega^v/t}\right|+$$

$$\sum_{\alpha=0}^n\frac{1}{\alpha!}\sum_a\sum_b\sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in M\\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\frac{\left[\rho_a\left(\overline{g}_b^a f_{b_{(\lambda_1,\lambda_2)}}^a\right)\right]}{t}\cdot\eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha)|(n,n)}e^{-\omega^v/t}\right|+$$

$$\sum_{\alpha=0}^{n-1}\frac{1}{\alpha!}\sum_a\sum_b\sum_k\sum_l\sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in M\\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\frac{\rho_0\mu_k d\overline{g}_l^k}{t}\frac{\left[\rho_a\left(\overline{g}_b^a f_{b_{(\lambda_1,\lambda_2)}}^a\right)\right]}{t}\cdot\eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha-1)|(n-1,n)}e^{-\omega^v/t}\right|+$$

$$\sum_{\alpha=0}^{n-1}\frac{1}{\alpha!}\sum_a\sum_b\sum_k\sum_l\sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in M\\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X\left|\left(\frac{-\sum_i\rho_i\left(\sum_{j=1}^n d\overline{g}_j^i\wedge dz_j\right)}{t}\right)^\alpha\frac{\rho_k d\overline{g}_l^k}{t}\frac{\left[\rho_a\left(\overline{g}_b^a f_{b_{(\lambda_1,\lambda_2)}}^a\right)\right]}{t}\cdot\eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha-1)|(n-1,n)}e^{-\omega^v/t}\right|.$$

And by 3.19, we have:

$$\leq \sum_{\alpha=0}^{n} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \cdot \eta_{(1\ldots n,1\ldots n)}^{(n-\alpha,n-\alpha)|(n,n)} \right| e^{-\delta/t} +$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \sum_k \sum_j \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_0 \mu_k d\overline{g}_j^k}{t} \cdot \eta_{(1\ldots n,1\ldots n)}^{(n-\alpha,n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t} +$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \sum_i \sum_j \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_i d\overline{g}_j^i}{t} \cdot \eta_{(1\ldots n,1\ldots n)}^{(n-\alpha,n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t} +$$

$$\sum_{\alpha=0}^{n} \frac{1}{\alpha!} \sum_a \sum_b \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in M \\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1,\lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha)|(n,n)} \right| e^{-\delta/t}$$

$$+ \sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \sum_a \sum_b \sum_k \sum_l \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in M \\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_0 \mu_k d\overline{g}_l^k}{t} \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1,\lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t} +$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \sum_a \sum_b \sum_k \sum_l \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in M \\ \lambda_1+\lambda_3=\lambda_2+\lambda_4=n}}$$

$$\int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_k d\overline{g}_l^k}{t} \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1,\lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3,\lambda_4)}^{(n-\alpha,n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t}.$$

Studying each parcel, we find constants $C_1 > 0, C_2 > 0, C_3 > 0, C_4 > 0, C_5 > 0, C_6 > 0$ such that:

$$\sum_{\alpha=0}^{n} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \cdot \eta_{(1\ldots n,1\ldots n)}^{(n-\alpha,n-\alpha)|(n,n)} \right| e^{-\delta/t} \leq$$

$$C_1 e^{-\delta/t} \left(1 + \frac{1}{t} + \cdots + \frac{1}{t^n} \right) Vol(X) \quad \stackrel{t \to 0}{\longrightarrow} 0;$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_0 \mu_k d\overline{g}_j^k}{t} \cdot \eta_{(1\ldots n,1\ldots n)}^{(n-\alpha,n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t} \leq$$

$$C_2 e^{-\delta/t} \left(\frac{1}{t} + \cdots + \frac{1}{t^n} \right) Vol(X) \quad \stackrel{t \to 0}{\longrightarrow} 0;$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_i d\overline{g}_j^i}{t} \cdot \eta_{(1 \dots n, 1 \dots n)}^{(n-\alpha, n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t} \leq$$
$$C_3 e^{-\delta/t} \left(\frac{1}{t} + \cdots + \frac{1}{t^n} \right) Vol(X) \quad \overset{t \to 0}{\longrightarrow} 0;$$

$$\sum_{\alpha=0}^{n} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1, \lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3, \lambda_4)}^{(n-\alpha, n-\alpha)|(n,n)} \right| e^{-\delta/t} \leq$$
$$C_4 e^{-\delta/t} \left(\frac{1}{t} + \cdots + \frac{1}{t^{n+1}} \right) Vol(X) \quad \overset{t \to 0}{\longrightarrow} 0;$$

$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_0 \mu_k d\overline{g}_l^k}{t} \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1, \lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3, \lambda_4)}^{(n-\alpha, n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t}$$
$$\leq C_5 e^{-\delta/t} \left(\frac{1}{t^2} + \cdots + \frac{1}{t^{n+1}} \right) Vol(X) \quad \overset{t \to 0}{\longrightarrow} 0;$$
$$\sum_{\alpha=0}^{n-1} \frac{1}{\alpha!} \int_X \left| \left(\frac{-\sum_i \rho_i \left(\sum_{j=1}^n d\overline{g}_j^i \wedge dz_j \right)}{t} \right)^\alpha \frac{\rho_k d\overline{g}_l^k}{t} \frac{\left[\rho_a \left(\overline{g}_b^a f_{b(\lambda_1, \lambda_2)}^a \right) \right]}{t} \cdot \eta_{(\lambda_3, \lambda_4)}^{(n-\alpha, n-\alpha-1)|(n-1,n)} \right| e^{-\delta/t}$$
$$\leq C_6 e^{-\delta/t} \left(\frac{1}{t^2} + \cdots + \frac{1}{t^{n+1}} \right) Vol(X) \quad \overset{t \to 0}{\longrightarrow} 0,$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are constants that limit superiorly the continuous functions in each integral (remember that X is a compact manifold). Since all the isolated parcels above go to zero when $t \to 0$, then the complete sums of each parcel go to zero too, and thus we get our result. □

In the next result, we will formalize the theorem of residues to an odd holomorphic supervector field V on a supermanifold S of $n|n$ dimension.

Theorem 3.41. Let V be an odd holomorphic supervector field on S with isolated singularities $p_i \in Sing(V)$, then for any $\eta \in \oplus A^{(p,q)|(r,s)}$ such that $(\overline{\partial} + i_V)(\eta) = 0$, we have:

$$\int_S \eta = \sum_i Res_{p_i}(V, \eta)$$

where

$$Res_{p_i}(V, \eta) = \lim_{t \to 0} \int_{SB_\epsilon(p_i)} \eta \cdot \exp \left\{ -\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t} \right\}$$

Proof. This proof is completely analogous to the proof of the theorem 3.16. Remember that here we consider only the points $p_i, p_j \in Sing(V)$ such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) =$

$\emptyset$ because to points $p_i, p_j \in Sing(V)$ such that $B_\epsilon(\mathbb{B}(p_i)) \bigcap B_\epsilon(\mathbb{B}(p_j)) \neq \emptyset$ we have $B_\epsilon(\mathbb{B}(p_i)) = B_\epsilon(\mathbb{B}(p_j))$, and consequently $SB_\epsilon(p_i) = SB_\epsilon(p_j)$ (look definition 3.31). □

3.5 Determination of residues for odd vectors

Now, we will determine, on some hypotheses, a formula to calculate the residues $Res_{p_i}(V, \eta)$ to odd vectors. First, consider the following lemma.

Lemma 3.42. Let $V = \sum_{i=1}^{m} f_i \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} g_j \frac{\partial}{\partial \xi_j}$ be an odd supervector field, in local coordinates, on the complex supermanifold S of $n|n$ dimension, where f_i are odd functions and g_j are even functions without odd variables. If $\det\left[\left(\mathbb{B}\left(\frac{\partial f_k}{\partial \xi_l}(p_j)\right)\right)_{n\times n}\right] \neq 0$ for some point $p_j \in S$, then:

$$\mathrm{Ber}(V)(p_j) = \frac{\det(JV)}{\det(D)}(p_j).$$

where

$$\mathrm{Ber}(V)(z;\xi) = \mathrm{sdet}\begin{pmatrix} \frac{\partial g_i}{\partial z_j} & \frac{\partial f_k}{\partial z_j} \\ \frac{\partial g_i}{\partial \xi_l} & \frac{\partial f_k}{\partial \xi_l} \end{pmatrix}, \quad JV = \left(\frac{\partial g_i}{\partial z_j}\right)_{n\times n}, \quad and \quad D = \left(\frac{\partial f_k}{\partial \xi_l}\right).$$

Proof. Consider $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with:

$$A = \left(\frac{\partial g_i}{\partial z_j}\right), \quad B = \left(\frac{\partial f_k}{\partial z_j}\right), \quad C = \left(\frac{\partial g_i}{\partial \xi_l}\right), \quad D = \left(\frac{\partial f_k}{\partial \xi_l}\right).$$

Now, note that $C = 0$ because the functions g_i do not have odd variables. Then, since $\det(\mathbb{B}(D))(p_j) \neq 0$ (hypothesis), by definition 1.97 we have:

$$\mathrm{Ber}(V)(p_j) = \left[\det(A - BD^{-1}C)\cdot(\det(D))^{-1}\right](p_j) = \frac{\det(A)}{\det(D)}(p_j) = \frac{\det(JV)}{\det(D)}(p_j).$$

□

With the notations of the definition 3.27 in hands, we have the following theorems.

Theorem 3.43. Let S be a compact complex supermanifold of $n|n$ dimension and let V be an odd holomorphic vector field with a non-degenerate isolated singularity $p_j \in Sing(V)$, whose representation in local coordinates is equal to $V = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial \xi_i}$, where $g_i(z)$ are even functions without odd variables and $f_i(z,\xi)$ are non-constant odd functions such that $f_i(p_j) = 0$. Furthermore, let ω be the form defined in 3.35 and let

$\eta \in \bigoplus A^{(p,q)|(r,s)}$ be a form such that $(\overline{\partial} + i_V)(\eta) = 0$. If the number n (dimension) is even (odd) and if the functions $\eta^{(0,0)|(n,n)}$, $\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ have only even (odd) quantities of variables ξ_j in its expansion, then:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j)$$

where

$$Ber(V) = \mathrm{sdet}\begin{pmatrix} \frac{\partial g_i}{\partial z_j} & \frac{\partial f_k}{\partial z_j} \\ \frac{\partial g_i}{\partial \xi_l} & \frac{\partial f_k}{\partial \xi_l} \end{pmatrix} \quad \textit{and} \quad D = \left(\tfrac{\partial f_k}{\partial \xi_l}\right).$$

Proof. Since $Res_{p_j}(V,\eta) = \lim_{t\to 0}\int_{SB_\epsilon(p_j)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\}$, then it's sufficient to show that:

$$\lim_{t\to 0}\int_{SB_\epsilon(p_j)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\} = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_j).$$

Then:

$$\int_{SB_\epsilon(p_j)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\} = \int_{SB_\epsilon(p_j)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t}\right\}\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\}.$$

By lemma 3.26, item (iii):

$$\begin{aligned}
&\int_{SB_\epsilon(p_j)} \eta\cdot\left\{\exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\} + \sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!}\frac{\left(\overline{\partial}\omega_1\right)^{(j-1)}\ \overline{\partial}\omega_2}{t^j}\right\}\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\} = \\
&\int_{SB_\epsilon(p_j)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega_1}{t}\right\}\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\} + \\
&\sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!}\int_{SB_\epsilon(p_j)} \eta\cdot\left(\frac{\left(\overline{\partial}\omega_1\right)^{(j-1)}\ \overline{\partial}\omega_2}{t^j}\right)\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\} = \\
&\sum_{k=0}^{n}\frac{(-1)^k}{k!}\int_{SB_\epsilon(p_j)} \eta\cdot\left(\frac{\overline{\partial}\omega_1}{t}\right)^k\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\} + \\
&\sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!}\int_{SB_\epsilon(p_j)} \eta\cdot\left(\frac{\left(\overline{\partial}\omega_1\right)^{(j-1)}\ \overline{\partial}\omega_2}{t^j}\right)\cdot\exp\left\{-\frac{i_V(\omega)}{t}\right\}.
\end{aligned}$$

Now, linearizing the functions near the singularity through the Taylor series, and doing the change of variable $z \mapsto z\sqrt{t}$, we have:

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_{SB_{\epsilon/\sqrt{t}}(p_j)} \eta \cdot \frac{\left(\overline{\partial}\omega_1\right)^k}{\sqrt{t}^{2k}} \cdot \exp\left\{-i_V(\omega)\right\} \det \begin{pmatrix} \sqrt{t} & & 0 \\ & \ddots & \\ 0 & & \sqrt{t} \end{pmatrix} +$$

$$\sum_{j=1}^{n} \frac{(-1)^j}{(j-1)!} \int_{SB_{\epsilon/\sqrt{t}}(p_j)} \eta \cdot \frac{\left(\overline{\partial}\omega_1\right)^{(j-1)} \ \overline{\partial}\omega_2}{\sqrt{t}^{2j}} \cdot \exp\left\{-i_V(\omega)\right\} \det \begin{pmatrix} \sqrt{t} & & 0 \\ & \ddots & \\ 0 & & \sqrt{t} \end{pmatrix} =$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_{SB_{\epsilon/\sqrt{t}}(p_j)} \eta \cdot \left(\overline{\partial}\omega_1\right)^k \cdot \exp\left\{-i_V(\omega)\right\} \cdot \sqrt{t}^{2n-2k} +$$

$$\sum_{j=1}^{n} \frac{(-1)^j}{(j-1)!} \int_{SB_{\epsilon/\sqrt{t}}(p_j)} \eta \cdot \left(\overline{\partial}\omega_1\right)^{(j-1)} \ \overline{\partial}\omega_2 \cdot \exp\left\{-i_V(\omega)\right\} \cdot \sqrt{t}^{2n-2j}.$$

Doing $t \to 0$, we have:

$$= \frac{(-1)^n}{n!} \int_{\mathbb{C}^{n|n}} \eta \cdot \left(\overline{\partial}\omega_1\right)^n \cdot \exp\left\{-i_V(\omega)\right\} + \frac{(-1)^n}{(n-1)!} \int_{\mathbb{C}^{n|n}} \eta \cdot \left(\overline{\partial}\omega_1\right)^{(n-1)} \ \overline{\partial}\omega_2 \cdot \exp\left\{-i_V(\omega)\right\}$$

By lemma 3.24, items (iv), (vi), and by definition 3.27, item (1), we have:

$$= \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \ \eta^{(0,0)|(n,n)} \cdot \exp\left\{-i_V(\omega)\right\} +$$

$$\frac{(-1)^n}{(n-1)!} \int_{\mathbb{C}^{n|n}} \sum_{j=1}^{n} (n-1)! \, (-1)^{\frac{n(n+1)}{2}} \left(dz_j \wedge d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \ \eta_{\widehat{j}}^{(1,0)|(n-1,n)} \right)$$

$$\wedge dz_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_j \wedge \cdots \wedge d\overline{g}_n \cdot \exp\left\{-i_V(\omega)\right\}.$$

Now, by lemma 3.29, we have:

$$= \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\{-i_V(\omega)\} +$$

$$(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \int_{\mathbb{C}^{n|n}} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$

$${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}$$

$$- (-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \int_{\mathbb{C}^{n|n}} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$

$${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\} =$$

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\{-i_V(\omega)\} +$$

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$

$${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}$$

$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$

$${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}.$$

To a better comprehension, let's study these integrals separately

(a)

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\{-i_V(\omega)\},$$

(b)

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \times$$

$$\int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}$$

$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \times$$

$$\int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}.$$

(a) By lemma 3.26, item (i), we have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\{-i_V(\omega)\} =$$

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\}$$
$$+ \quad \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$\eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \left(-\sum \overline{g}_i f_i\right). \quad (3.22)$$

Let's calculate the first integral of (3.22):

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \, \eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} = \quad (3.23)$$

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)} (p_j) \int_{\mathbb{C}^n} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \; \exp\left\{-\sum \overline{g}_i g_i\right\} =$$
$$\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)} (p_j) \int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \; \exp\left\{-\sum \overline{g}_i g_i\right\} =$$
$$\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)} (p_j) \times$$
$$\int_{\mathbb{R}^{2n}} det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial y_1} & & & \\ \frac{\partial \overline{g}_1}{\partial x_1} & \frac{\partial \overline{g}_1}{\partial y_1} & & & \\ & & \ddots & & \\ & & & \frac{\partial g_n}{\partial x_n} & \frac{\partial g_n}{\partial y_n} \\ & & & \frac{\partial \overline{g}_n}{\partial x_n} & \frac{\partial \overline{g}_n}{\partial y_n} \end{pmatrix} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \; \exp\left\{-(x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2)\right\} \quad (3.24)$$

But since

$$det\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial y_1} & & & \\ \frac{\partial \overline{g}_1}{\partial x_1} & \frac{\partial \overline{g}_1}{\partial y_1} & & & \\ & & \ddots & & \\ & & & \frac{\partial g_n}{\partial x_n} & \frac{\partial g_n}{\partial y_n} \\ & & & \frac{\partial \overline{g}_n}{\partial x_n} & \frac{\partial \overline{g}_n}{\partial y_n} \end{pmatrix} =$$

$$det\begin{pmatrix} \frac{\partial}{\partial x_1}(x_1+iy_1) & \frac{\partial}{\partial y_1}(x_1+iy_1) & & & \\ \frac{\partial}{\partial x_1}(x_1-iy_1) & \frac{\partial}{\partial y_1}(x_1-iy_1) & & & \\ & & \ddots & & \\ & & & \frac{\partial}{\partial x_n}(x_n+iy_n) & \frac{\partial}{\partial y_n}(x_n+iy_n) \\ & & & \frac{\partial}{\partial x_n}(x_n-iy_n) & \frac{\partial}{\partial y_n}(x_n-iy_n) \end{pmatrix} =$$

$$det\begin{pmatrix} 1 & i & & & \\ 1 & -i & & & \\ & & \ddots & & \\ & & & 1 & i \\ & & & 1 & -i \end{pmatrix} = (-2i)^n \, ,$$

then we have that the equation (3.24) is equal to:

$$\frac{\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{det(JV)}(p_j)\int_{\mathbb{R}^{2n}}(-2i)^n dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n\cdot\exp\left\{-x_1^2-y_1^2-\cdots-x_n^2-y_n^2\right\} =$$

$$\frac{(-2i)^n\cdot\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{det(JV)}(p_j)\int_{-\infty}^{+\infty}\exp\left\{-x_1^2\right\}dx_1\int_{-\infty}^{+\infty}\exp\left\{-y_1^2\right\}dy_1\cdots$$

$$\cdots\int_{-\infty}^{+\infty}\exp\left\{-x_n^2\right\}dx_n\int_{-\infty}^{+\infty}\exp\left\{-y_n^2\right\}dy_n =$$

$$\frac{(-2i)^n\cdot\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{det(JV)}(p_j)\left(\sqrt{\pi}\right)^{2n} = (-2\pi i)^n\frac{\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{det(JV)}(p_j) = \left(\frac{2\pi}{i}\right)^n\frac{\eta^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{det(JV)}(p_j)\,.$$

To analyze the second integral of (3.22), the hypotheses will be necessary:

- (Even case) The second Berezin integral is equal to zero because $\overline{g}_i f_i$ are odd functions (odd quantities of variables ξ_j) but, by hypothesis, $\eta^{(0,0)|(n,n)}$ has only even quantities of variables ξ_j. Furthermore (by hypothesis) n is an even number (even dimension). Therefore, we will never find an even top quantity of variables ξ_j, i.e, we will always find the following:

$$\underbrace{d\xi_1 \wedge \cdots \wedge d\xi_n}_{\text{even quantity}} \underbrace{\xi_{j_1} \cdots \xi_{j_{(2l+1)}}}_{\text{odd quantity}} \xrightarrow{\int} 0. \tag{3.25}$$

- (Odd case) The second Berezin integral is equal to zero because $\overline{g}_i f_i$ are odd functions (odd quantities of variables ξ_j) but, by hypothesis, $\eta^{(0,0)|(n,n)}$ has only odd quantities of variables ξ_j. Furthermore (by hypothesis) n is an odd number (odd dimension). Therefore, we will never find an odd top quantity of variables ξ_j, i.e, we will always find the following:

$$\underbrace{d\xi_1 \wedge \cdots \wedge d\xi_n}_{\text{odd quantity}} \underbrace{\xi_{j_1} \cdots \xi_{j_{2l}}}_{\text{even quantity}} \xrightarrow{\int} 0. \tag{3.26}$$

Then:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots$$
$$\cdots \wedge d\overline{\xi}_n \ \eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \left(-\sum \overline{g}_i f_i\right) = 0 \tag{3.27}$$

Therefore:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \ \eta^{(0,0)|(n,n)} \cdot \exp\{-i_V(\omega)\} =$$
$$\left(\frac{2\pi}{i}\right)^n \frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)} (p_j)\,. \tag{3.28}$$

(b)

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}$$
$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\{-i_V(\omega)\}$$

And by lemma 3.26, item (i):

$$= \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}(p_j)\int_{\mathbb{C}^n} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\ \exp\left\{-\sum \overline{g}_i g_i\right\}$$
$$-\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \; {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}(p_j)\int_{\mathbb{C}^n} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\ \exp\left\{-\sum \overline{g}_i g_i\right\}$$
$$-\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\xi_n\wedge d\overline{\xi}_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\left(\sum \overline{g}_j f_j\right)$$
$$+\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\xi_n\wedge d\overline{\xi}_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\left(\sum \overline{g}_j f_j\right).$$

Since ${}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ is an even function and as ${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}$ is an odd function (look remark 3.28), we have:

$$= \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}(p_j)\int_{\mathbb{C}^n} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\ \exp\left\{-\sum \overline{g}_i g_i\right\}$$
$$+\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\xi_n\wedge d\overline{\xi}_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$${}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\left(\sum \overline{g}_j f_j\right). \quad (3.29)$$

To the first integral of (3.29), we have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}} \; {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}(p_j)\int_{\mathbb{C}^n} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\ \exp\left\{-\sum \overline{g}_i g_i\right\} =$$

$$\left(\frac{2\pi}{i}\right)^n\left[\frac{{}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}\right](p_j) = \left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\dots n,1\dots n)}}{det\,(JV)}\right](p_j)\,. \quad (3.30)$$

And by a similar argument showed in (3.25) or (3.26) (assuming the hypotheses), we have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_j)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots$$
$$\cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\}\left(\sum \overline{g}_j f_j\right) = 0.$$

Therefore, by equations (3.28) and (3.30), we have:

$$\lim_{t\to 0} \int_{SB_\epsilon(p_j)} \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\} = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{det\,(JV)}\right](p_j)\,.$$

Since $p_j \in Sing(V)$ is non-degenerate, by lemma 3.42 we conclude that:

$$\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{det\,(JV)}\right](p_j) = \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_j)\,.$$

Then

$$\lim_{t\to 0} \int_{SB_\epsilon(p_j)} \eta \cdot \exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\} = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_j)\,.$$

Therefore

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_j)\,.$$

□

Corollary 3.44. If on the theorem 3.43 we add the condition $\det(\mathbb{B}(D))(\mathbb{B}(p_j)) = 1$, then we have:

$$Res_{p_j}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)}\right](p_j)\,.$$

In the previous theorem, we imposed conditions over the form η and over the supermanifold's dimension. Now we go to choose a class of odd functions such that those hypotheses will not be more necessary.

Theorem 3.45. Let S be a compact complex supermanifold of $n|n$ dimension and let V be an odd holomorphic vector field with a non-degenerate isolated singularity $p_\kappa \in Sing(V)$, whose representation in local coordinates is equal to $V = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial \xi_i}$, where $g_i(z)$ are even functions without odd variables and $f_i(z,\xi)$ are odd functions such that $f_i(z,\xi) = \sum_{\lambda\in M}\ \xi^\lambda \cdot a^i_\lambda \cdot g_i + \sum_{\lambda\in M}\ \xi^\lambda \cdot b^i_\lambda \cdot g_{i_\lambda}$ with M being the set of multi-indices, $a^i_\lambda, b^i_\lambda \in \mathbb{C}_S$ and $i_\lambda \in \{1,\ldots,\widehat{i},\ldots,n\}$. Furthermore, let ω be the form defined in 3.35 and let $\eta \in \bigoplus A^{(p,q)|(r,s)}$ be a form such that $(\overline{\partial} + i_V)(\eta) = 0$. Then, we have:

$$Res_{p_\kappa}(V,\eta) = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_\kappa) \ + \left(\frac{2\pi}{i}\right)^n \left[\frac{\sum_{j=1}^n \sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)} \left\{a^j_\lambda \left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)} - \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_\kappa)\,.$$

where $L(\lambda)$ are odd numbers and

$$Ber(V) = \mathrm{sdet}\begin{pmatrix} \frac{\partial g_i}{\partial z_j} & \frac{\partial f_k}{\partial z_j} \\ \frac{\partial g_i}{\partial \xi_l} & \frac{\partial f_k}{\partial \xi_l}\end{pmatrix} \qquad and \qquad D = \left(\frac{\partial f_k}{\partial \xi_l}\right).$$

Proof. Since $Res_{p_\kappa}(V,\eta) = \lim_{t\to 0}\int_{SB_\epsilon(p_\kappa)} \eta\cdot \exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\}$, then it's sufficient to show that:

$$\lim_{t\to 0}\int_{SB_\epsilon(p_\kappa)} \eta\cdot \exp\left\{-\frac{\overline{\partial}\omega}{t} - \frac{i_V(\omega)}{t}\right\} = \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)} + \eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_\kappa) \ + \left(\frac{2\pi}{i}\right)^n \left[\frac{\sum_{j=1}^n \sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)} \left\{a^j_\lambda \left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)} - \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)\cdot \det(D)}\right](p_\kappa)\,.$$

Following the same steps of the theorem 3.43, we find:

(a)

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\times \int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ \eta^{(0,0)|(n,n)}\cdot\exp\left\{-i_V(\omega)\right\},$$

(b)

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}}\times$$
$$\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\{-i_V(\omega)\}$$
$$-\ \frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}}\times$$
$$\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\{-i_V(\omega)\}\,.$$

(a) By lemma 3.26, item (i), we have:

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ \eta^{(0,0)|(n,n)}\cdot\exp\{-i_V(\omega)\}=$$
$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ \eta^{(0,0)|(n,n)}\cdot\exp\left\{-\sum\overline{g}_ig_i\right\}\ +$$
$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$$\eta^{(0,0)|(n,n)}\cdot\exp\left\{-\sum\overline{g}_ig_i\right\}\left(-\sum\overline{g}_if_i\right).$$

The first integral is equal to:

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ \eta^{(0,0)|(n,n)}\cdot\exp\left\{-\sum\overline{g}_ig_i\right\}$$
$$=\left(\frac{2\pi}{i}\right)^n\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)}(p_\kappa)\,.\qquad(3.31)$$

Now, let's calculate the second integral:

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$$\eta^{(0,0)|(n,n)}\cdot\exp\left\{-\sum\overline{g}_ig_i\right\}\left(-\sum\overline{g}_jf_j\right)=$$

$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \sum_j \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$\eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j f_j =$$

$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$\left(\xi^\mu \cdot \overline{\xi}^1 \cdots \overline{\xi}^n \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right) \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \xi^\lambda \cdot \left(a^j_\lambda g_j + b^j_\lambda g_{j_\lambda}\right) =$$
$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$\left(\xi^\mu \cdot \overline{\xi}^1 \cdots \overline{\xi}^n \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right) \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \xi^\lambda \cdot a^j_\lambda g_j +$$
$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$\left(\xi^\mu \cdot \overline{\xi}^1 \cdots \overline{\xi}^n \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right) \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \xi^\lambda \cdot b^j_\lambda g_{j_\lambda} =$$
$$- \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{a^j_\lambda \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)} (p_\kappa) \int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_j$$
$$- \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{b^j_\lambda \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)} (p_\kappa) \int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_{j_\lambda}. \tag{3.32}$$

where λ, μ are multi-indices (λ is an odd multi-index because f is odd), M is the set of all multi-indices and L is the function that gives the length of the multi-index.

Then, from (3.32), we have the two parcels:

$$- \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{a^j_\lambda \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)} (p_\kappa) \int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_j, \tag{3.33}$$

and

$$- \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{b^j_\lambda \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)} (p_\kappa) \int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_{j_\lambda}. \tag{3.34}$$

First, let's calculate the integral from (3.34):

$$\int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_{j_\lambda} =$$
$$\int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n (-2i)^n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} (x_j - iy_j)(x_{j_\lambda} + iy_{j_\lambda}).$$

Since $(x_j - iy_j)(x_{j_\lambda} + iy_{j_\lambda}) = x_j x_{j_\lambda} + ix_j y_{j_\lambda} - ix_{j_\lambda} y_j + y_j y_{j_\lambda}$, then we have:

$$= (-2i)^n \int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} \cdot x_j x_{j_\lambda} +$$
$$i(-2i)^n \int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} \cdot x_j y_{j_\lambda} -$$
$$i(-2i)^n \int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} \cdot x_{j_\lambda} y_j +$$
$$(-2i)^n \int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} \cdot y_j y_{j_\lambda} =$$

$$(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-x_j^2\} x_j}_{=0} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-x_{j_\lambda}^2\} x_{j_\lambda}}_{=0} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} +$$
$$i(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-x_j^2\} x_j}_{=0} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-y_{j_\lambda}^2\} y_{j_\lambda}}_{=0} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} -$$
$$i(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-x_{j_\lambda}^2\} x_{j_\lambda}}_{=0} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-y_j^2\} y_j}_{=0} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} +$$
$$(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-y_j^2\} y_j}_{=0} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-y_{j_\lambda}^2\} y_{j_\lambda}}_{=0} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} = 0.$$

Thus:

$$-\sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{b_\lambda^j \cdot \eta_{(\mu,1\ldots n)}^{(0,0)|(n,n)}}{det(JV)} (p_\kappa) \times$$
$$\int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_{j_\lambda} = 0. \quad (3.35)$$

with $j_\lambda \in \{1, \ldots, \widehat{j}, \ldots, n\}$.

Now, let us to calculate the integral from (3.33):

$$\int_{\mathbb{C}^n} dg_1 \wedge d\overline{g}_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_n \;\exp\left\{-\sum \overline{g}_i g_i\right\} \cdot \overline{g}_j \cdot g_j =$$
$$\int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n (-2i)^n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\}(x_j^2 + y_j^2) =$$
$$\int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n (-2i)^n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} x_j^2 +$$
$$\int_{\mathbb{R}^{2n}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n (-2i)^n \cdot \exp\left\{-x_1^2 - y_1^2 - \cdots - x_n^2 - y_n^2\right\} y_j^2 =$$

$$(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-x_j^2\} x_j^2}_{=\frac{\sqrt{\pi}}{2}} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} +$$
$$(-2i)^n \int_{-\infty}^{\infty} \exp\{-x_1^2\} \cdots \underbrace{\int_{-\infty}^{\infty} \exp\{-y_j^2\} y_j^2}_{=\frac{\sqrt{\pi}}{2}} \cdots \int_{-\infty}^{\infty} \exp\{-y_n^2\} =$$
$$(-2i)^n \cdot (\sqrt{\pi})^{2n-1} \cdot \frac{\sqrt{\pi}}{2} + (-2i)^n \cdot (\sqrt{\pi})^{2n-1} \cdot \frac{\sqrt{\pi}}{2} = (-2\pi i)^n = \left(\frac{2\pi}{i}\right)^n . \quad (3.36)$$

So, by (3.32), (3.33), (3.35) and (3.36), we have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots$$
$$\cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \cdots d\overline{\xi}_n \;\eta^{(0,0)|(n,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\}\left(-\sum \overline{g}_j f_j\right) =$$
$$-\left(\frac{2\pi}{i}\right)^n \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{a_\lambda^j \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)} (p_\kappa) . \quad (3.37)$$

Therefore, by (3.31) and (3.37), we finally have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \;\eta^{(0,0)|(n,n)} \cdot \exp\left\{-i_V(\omega)\right\} =$$
$$\left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}}{det(JV)} - \sum_j \sum_{(\lambda,\mu \in M | L(\lambda)+L(\mu)=n)} \frac{a_\lambda^j \cdot \eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}}{det(JV)}\right] (p_\kappa) . \quad (3.38)$$

(b)Let us to calculate:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$ {}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)} \cdot \exp\{-i_V(\omega)\}$$
$$- \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$ {}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)} \cdot \exp\{-i_V(\omega)\} =$$

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}\ {}^{E}\eta_{\widehat{*}(1\ldots n,1\ldots n)}^{(1,0)|(n-1,n)}}{det\,(JV)} (p_\kappa) \int_{\mathbb{C}^{n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} -$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}\ {}^{O}\eta_{\widehat{*}(1\ldots n,1\ldots n)}^{(1,0)|(n-1,n)}}{det\,(JV)} (p_\kappa) \int_{\mathbb{C}^{n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} -$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$ {}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \left(\sum \overline{g}_j f_j\right) +$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\overline{\xi}_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$ {}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \left(\sum \overline{g}_j f_j\right).$$

And since ${}^{E}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ is an even function and ${}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)}$ is an odd function (look remark 3.28), we have:

$$= \frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}\ {}^{E}\eta_{\widehat{*}(1\ldots n,1\ldots n)}^{(1,0)|(n-1,n)}}{det\,(JV)} (p_\kappa) \int_{\mathbb{C}^{n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \ \exp\left\{-\sum \overline{g}_i g_i\right\} +$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}} \int_{\mathbb{C}^{n|n}} dg_1 \wedge \cdots \wedge dg_n \wedge d\overline{g}_1 \wedge \cdots \wedge d\overline{g}_n \wedge d\xi_1 \wedge \cdots \wedge d\overline{\xi}_n \times$$
$$ {}^{O}\eta_{\widehat{*}}^{(1,0)|(n-1,n)} \cdot \exp\left\{-\sum \overline{g}_i g_i\right\} \left(\sum \overline{g}_j f_j\right). \quad (3.39)$$

Then, by the first parcel of (3.39):

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}\ {}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{det\,(JV)}(p_\kappa)\int_{\mathbb{C}^n} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\ \exp\left\{-\sum \overline{g}_i g_i\right\} =$$
$$\left(\frac{2\pi}{i}\right)^n \left[\frac{{}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{det\,(JV)}\right](p_\kappa) \;=\; \left(\frac{2\pi}{i}\right)^n \left[\frac{\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{det\,(JV)}\right](p_\kappa)\,. \quad (3.40)$$

Now, by the second parcel of (3.39), we have:

$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots$$
$$\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\left(\sum \overline{g}_j f_j\right) =$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$$\left(\xi^\mu\cdot\overline{\xi}^1\cdots\overline{\xi}^n\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}\right)\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\cdot\overline{g}_j\xi^\lambda\cdot\left(a^j_\lambda g_j + b^j_\lambda g_{j_\lambda}\right) =$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$$\left(\xi^\mu\cdot\overline{\xi}^1\cdots\overline{\xi}^n\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}\right)\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\cdot\overline{g}_j\xi^\lambda\cdot a^j_\lambda\cdot g_j +$$
$$\frac{(-1)^n \cdot (-1)^{\frac{n(n+1)}{2}}}{det(JV)_{(p_\kappa)}}\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\times$$
$$\left(\xi^\mu\cdot\overline{\xi}^1\cdots\overline{\xi}^n\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}\right)\cdot\exp\left\{-\sum \overline{g}_i g_i\right\}\cdot\overline{g}_j\xi^\lambda\cdot b^j_\lambda\cdot g_{j_\lambda}. \quad (3.41)$$

And by analogous calculations to those made previously, we have that equation (3.41) is equal to

$$\left(\frac{2\pi}{i}\right)^n \sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\frac{a^j_\lambda\cdot\ {}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}}{det(JV)}(p_\kappa) =$$
$$\left(\frac{2\pi}{i}\right)^n \sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\frac{a^j_\lambda\cdot\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}}{det(JV)}(p_\kappa) \quad (3.42)$$

Therefore, by equations (3.40) and (3.42), we have:

$$\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\;{}^{E}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\{-i_V(\omega)\}$$
$$-\frac{(-1)^n\cdot(-1)^{\frac{n(n+1)}{2}}}{det\,(JV)_{(p_\kappa)}}\int_{\mathbb{C}^{n|n}} dg_1\wedge\cdots\wedge dg_n\wedge d\overline{g}_1\wedge\cdots\wedge d\overline{g}_n\wedge d\xi_1\wedge\cdots\wedge d\overline{\xi}_n\;{}^{O}\eta^{(1,0)|(n-1,n)}_{\widehat{*}}\cdot\exp\{-i_V(\omega)\}$$
$$=\left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{det\,(JV)}+\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\frac{a^j_\lambda\cdot\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}}{det(JV)}\right](p_\kappa)\quad(3.43)$$

Then, by equations (3.38) and (3.43), we have:

$$\lim_{t\to 0}\int_{SB_\epsilon(p_\kappa)}\eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t}-\frac{i_V(\omega)}{t}\right\}=$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{det\,(JV)}\right](p_\kappa)\;+$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{det\,(JV)}\right](p_\kappa)$$

Since $p_\kappa\in Sing(V)$ is non-degenerate, by lemma 3.42 we conclude that:

$$\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{det\,(JV)}\right](p_\kappa)=\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n;1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n;1\ldots n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\,.$$

and

$$\left[\frac{\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{det\,(JV)}\right](p_\kappa)=$$
$$\left[\frac{\sum_j\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\,.$$

Then:

$$\lim_{t\to 0}\int_{SB_\epsilon(p_\kappa)} \eta\cdot\exp\left\{-\frac{\overline{\partial}\omega}{t}-\frac{i_V(\omega)}{t}\right\} =$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\;+$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\sum_{j=1}^n\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\,.$$

Therefore:

$$Res_{p_\kappa}(V,\eta)=\left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\;+$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\sum_{j=1}^n\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)\cdot\det(D)}\right](p_\kappa)\,.$$

□

Corollary 3.46. If on the theorem 3.45 we add the condition $\det(\mathbb{B}(D))(\mathbb{B}(p_\kappa)) = 1$, then we have:

$$Res_{p_\kappa}(V,\eta)=\left(\frac{2\pi}{i}\right)^n\left[\frac{\eta^{(0,0)|(n,n)}_{(1\ldots n,1\ldots n)}+\eta^{(1,0)|(n-1,n)}_{\widehat{*}(1\ldots n,1\ldots n)}}{\mathrm{Ber}(V)}\right](p_\kappa)\;+$$
$$\left(\frac{2\pi}{i}\right)^n\left[\frac{\sum_{j=1}^n\sum_{(\lambda,\mu\in M|L(\lambda)+L(\mu)=n)}\left\{a^j_\lambda\left(\eta^{(1,0)|(n-1,n)}_{\widehat{*}(\mu,1\ldots n)}-\eta^{(0,0)|(n,n)}_{(\mu,1\ldots n)}\right)\right\}}{\mathrm{Ber}(V)}\right](p_\kappa)\,.$$

where $L(\lambda)$ are odd numbers.

3.6 Examples

3.6.1 Duistermaat-Heckman type formula

Let V be a vector field with only nondegenerate zero components on a supermanifold of dimension $n|m$. If ω is a 2-form of type $(1,1)$ such that $\omega^n \neq 0$, $(\overline{\partial}+i_V)\omega = 0$ and there is a smooth superfunction g such that $i_V\omega = \overline{\partial}(g)$, then for any $s > 0$, then under the conditions of Theorems proved in this Thesis we obtain the following Duistermaat-Heckman type Formula

$$\int_X e^{-sg}\frac{\omega^n}{n!} = \left(\frac{2\pi}{i}\right)^n \sum_{p_\kappa \in Sing(V)} \left[\frac{(e^{\omega - sg})^{(0,0)|(n,n)}_{(1\dots n,1\dots n)}}{\mathrm{Ber}(V)}\right](p_\kappa)$$

3.6.2 Complex projective superspaces

The complex projective superspace $\mathbb{P}^{n|m}$ is the supermanifold obtained as the quotient of $\mathbb{C}^{n+1|m}$ by the $\mathbb{C}^*$-action which is defined as

$$\lambda \cdot (z^0, \dots, z^n, \zeta^1, \dots, \zeta^m) := (\lambda z^0, \dots, \lambda z^n, \lambda\zeta^1, \dots, \lambda\zeta^m)$$

for all $\lambda \in \mathbb{C}^*$. See [14] for more details about projective superspace.

Consider the Kähler supermetric on $\mathbb{P}^{n|m}$ given by the Fubini-Study supermetric:

$$\omega_{FS}([Z]) = \frac{i}{2\pi}\partial\bar{\partial}\log|Z|^2 = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + |Z_0|^2),$$

where $|Z_0|^2 = 1 + \sum_i \bar{z}_0^i z_0^i + i\sum_\iota \bar{\zeta}_0^\iota \zeta_0^\iota$. On $U_0 = \{z^0 \neq 0\} \subset \mathbb{P}^{m|n}$ we have the local coordinates $(Z_0^I) = (z_0^i, \zeta_0^\iota)$, $I = 1, \dots, m+n$, where $z_0^i = \frac{z^i}{z^0}$ and $\zeta_0^\iota = \frac{\zeta^\iota}{z^0}$. Thus,

$$\omega_{FS}|_{U_0} = \sum_{I,J} \omega_{IL}\, dZ_0^I \wedge dZ_0^L$$

with

$$\omega_{IL} = \left(\begin{array}{c|c} \omega_{il} & \omega_{i\lambda} \\ \hline \omega_{\iota l} & \omega_{\iota\lambda} \end{array}\right) = \frac{i}{2\pi(1+|Z_0|^2)^2}\left(\begin{array}{c|c} \delta^{il}(1+|Z_0|^2) - \bar{z}_0^i z_0^l & -i\bar{z}_0^i\zeta_0^\lambda \\ \hline i\bar{\zeta}_0^\iota z_0^l & i\delta^{\iota\lambda}(1+|Z_0|^2) - \bar{\zeta}_0^\iota\zeta_0^\lambda \end{array}\right).$$

If $n \geq m$, then It follows from [9, pg. 28] that

$$\int_{\mathbb{P}^{n|m}} \omega_{FS}^n = \frac{1}{(n-m)!},$$

By the same computation in the case $m = 0$, we can conclude that $(\overline{\partial} + i_V)\omega_{FS}^n = 0$ for all vector field V on $\mathbb{P}^{n|m}$, see for instance [1, pages 25-26]. Therefore, by Theorem 3.15 and Theorem 3.40 ($n = m$) we conclude that projective superspace $\mathbb{P}^{n|m}$, with $n \geq m$, has no vector field without singularities.

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