When Bob cannot trust Alice

A semi-device-independent tale of quantum steering

Master's Thesis

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"Of course it is happening inside your head, Harry,
but why on earth should that mean that it is not real?"

— J. K. Rowling

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Abstract

Einstein-Podolsky-Rosen steering is a form of quantum correlation intermediate between entanglement and Bell nonlocality. Given that Alice and Bob share a quantum state, we say Alice can steer Bob when it is possible for her to condition Bob's state by interacting only with her subsystems, in such a way that the resulting correlation between her inputs, her outcomes, and Bob's conditioned states exhibits nonlocal properties. In this dissertation, we study the phenomenon of quantum steering from a fundamental perspective and as a semi-device-independent certification of entanglement. We formalize steering by making comparisons with other well studied quantum correlations and by investigating the hierarchy between them. We explore the close relationship between steering and joint measurability and present the phenomenon of one-way steering. The main focus of this work is the problem of determining the steerability of quantum states subjected to restricted measurement scenarios. In each scenario, we search for the family of measurements that reveals highest steerability of a given quantum state. We develop and apply a variety of methods for performing this task that are based on semidefinite programming. Analytical solutions for a particular family of measurements are achieved by exploring the theory of joint measurability and constructing steering inequalities. Finally, we study the relevance of general POVMs in exhibiting steering when compared to projective measurements.

Resumo

Einstein-Podolsky-Rosen *steering* é uma forma de correlação quântica intermediária entre emaranhamento e não-localidade de Bell. Dado que Alice e Bob compartilham um estado quântico, diz-se que Alice pode *steer* Bob quando é possível que ela condicione o estado de Bob interagindo somente com os seus próprios subsistemas, de tal forma que a correlação resultante entre seus *inputs*, seus resultados e os estados condicionados de Bob apresente propriedades não-locais. Nesta dissertação, nós estudamos o fenômeno de *steering* quântico de um ponto de vista fundamental e como uma forma semi-independente de dispositivo de certificação de emaranhamento. Formalizamos *steering* por meio de comparações com outras correlações quânticas bem estudadas e investigando a hierarquia entre elas. Exploramos a relação próxima entre *steering* e *joint measuralibility* e apresentamos o fenômeno de *one-way steering*. O principal foco desse trabalho é o problema de determinar a *steerability* de estados quânticos sujeitos a cenários de medições restritos. Em cada cenário, procuramos pela família de medições que revela a maior *steerability* de um dado estado quântico. Desenvolvemos e aplicamos uma variedade de métodos para executar essa tarefa que são baseados em programação semidefinida. Soluções analíticas para uma família particular de medições são encontradas utilizando a teoria de *joint measurability* e pela construção de desigualdades de *steering*. Finalmente, estudamos a relevância de POVMs gerais em exibir *steering* quando comparados a medições projetivas.

Contents

Introduction

This is a dissertation about steering. So I guess it would be fair for you to ask me what steering is. I could answer:

Steering is a form of quantum correlation.

Chances are, you are not going to like that. I mean, it is correct. But the term quantum correlations, though often used in quantum information texts, can be somewhat vague (or obscure).

Then, I could try and repeat to you the usual story from abstracts and introductions of texts about steering[1]:

> *Steering is a form of quantum correlation intermediate between entanglement and Bell nonlocality. It was first introduced by Schrödinger [1], in an attempt to formalize the apparent Einstein-Podolsky-Rosen paradox [2], as the ability of one party to affect the state of the other by her choice of measurement basis, and recently defined as an operational task by Wiseman, Jones, and Doherty [3].*

If you know a bit about Bell nonlocality and entanglement this might have helped. Otherwise, you are probably lacking some intuition on the matter. We could try an operational description, like a task performed by Alice and Bob.

[1] Including this one.

> *[Suppose] Alice can prepare a bipartite quantum state and send one part to Bob and repeat this any number of times. Each time, they measure their respective parts and communicate classically. Alice's task is to convince Bob that the state she can prepare is entangled. Bob [...] accepts that quantum mechanics describes the results of the measurements he makes. However, Bob does not trust Alice. If the correlations between Bob's measurement results and the results Alice reports can be explained [classically] then Bob will not be convinced that the state is entangled; Alice could have drawn a pure state at random from some ensemble and sent it to Bob, and then chosen her result based on her knowledge [...]. Conversely, if the correlations cannot be so explained then the state must be entangled. Therefore, Alice will succeed in her task if, and only if, she can create genuinely different ensembles for Bob by* ***steering*** *Bob's state.*[2]

We say their shared state is *steerable* when it is possible for Alice to *steer* Bob. When this is not possible, we call their state *unsteerable*.

But is this information enough to allow one to fully grasp the concept of quantum steering? At this point, I suspect there are still many doubts about what steering means and what this phenomenon has to say about our classical view of the world. After all, we still lack technical details in order to be truly able to understand both the fundamental value of steering in the theory of quantum mechanics and its applications to quantum information tasks.

I hope this dissertation can bring you closer to the answer.

In chapter 1 we present an introduction to entanglement, steering, and Bell nonlocality. Our approach in this chapter is to define and compare the central objects in the study of these phenomena in order to explore the relationship between them. We discuss the

[2] Extracted from H. M. Wiseman, S. J. Jones and, A. C. Doherty. Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, *Phys. Rev. Lett.* **98**, 140402 (2007).

assumption of device dependence in each scenario and present an information theoretic task in which Alice and Bob certify entanglement by means of a steering test.

In chapter 2 we deepen the study of the theory of quantum steering by presenting some important results on the literature of the subject. They include the phenomenon of one-way steering, the role of local operations on steering tasks, its tight connection to joint measurability, and the quantification of *steerability* – the ability of exhibiting steering.

In chapter 3 we characterize some important steering problems with semidefinite programming. In this chapter, we begin to introduce the original material of this dissertation by presenting four different methods for deciding whether or not quantum states are steerable when subjected to a finite number of measurements.

Finally, chapter 4 is dedicated to the presentation of our novel results. They involve: upper and lower bounds for the steerability of qubit Werner states in scenarios of fixed number of measurements; an investigation of the optimal set of measurements for steering on each scenario; analytical solutions for a particular family of measurements that were based on duality theory of convex optimization and achieved by explicitly providing local hidden state models and steering inequalities; and an analysis of the relevance of general POVMs in exhibiting steering when compared to projective measurements.

CHAPTER 1

Entanglement, EPR Steering, and Bell Nonlocality

This dissertation begins by introducing three forms of correlations exhibited by physical systems that are predicted by quantum theory and defy our classical view of the world. We will define and discuss important aspects of entanglement and Bell nonlocality in such a way that they can be compared to analogous concepts of quantum steering, the main subject of this work.

On this chapter, some key concepts of quantum mechanics that will be recurrently used throughout this text will be introduced. However, the intention is not to provide a review nor an introduction of quantum theory. For those, we refer the reader to the book in reference [4] and to the lecture notes on reference [5], both focused on quantum information.

The goals of this chapter are to familiarize the reader with the approaches and scenarios that are most useful to the study of these quantum correlations, to help in the understanding of the similarities, differences, and dependences between them, and hopefully to assist the reader on realizing how these correlations differ from our everyday intuition.

1.1 Device and semi-device independence

Before introducing entanglement, steering, and nonlocality, it is important to clarify the two main approaches we will use to study these correlations. The approach chosen for each situation will tell us how to handle the information that is accessible under those circumstances.

The two main approaches used in this text are the device-dependent approach and the device-independent approach. The typical situation of interest is the one where two parties, called Alice and Bob, are involved. Each of them can be considered to be in their own laboratories, dealing with their own experiments and experimental setups.

If one choses to use the **device-dependent** approach, then the parties involved are considered to have full control over their setups. This means that it is possible to assume that both Alice and Bob know the physical systems they are handling, the measurements they can perform and all the details about their experimental apparatus, such as efficiency and possible flaws. When using this approach, the parties are often said to be **trusted**. They have access to all the information about their experiments that they desire.

On the other hand, if one chooses the **device-independent** approach, the circumstances are opposite. In this case, Alice and Bob are considered to have very limited access to their experiments and to the information that can be yielded by them. Both parties are considered to be dealing with **black boxes**. A black box can be thought of as an actual box with buttons that one can choose to press in any order. For each choice of button (input), the box will return an answer (output). The idea behind a black box is that it is not known, and it is not possible to figure out, what is the mechanism that governs the box's behavior. It is not necessary to assume the experimenters know any details about their setups - or even whether or not they are dealing with physical systems. As far as they are concerned, the boxes could be filled with fairies and unicorns, as they will only choose which buttons to press and look at the outcomes. The information that is accessible to Alice and Bob are the probability distributions of getting certain outcomes when choosing certain inputs of the box. In this approach, the parties are often called **unstrusted**, or considered to have received their experimental setups from an untrusted third party.

Now let us use this approach to study some quantum correlations.

In the typical entanglement scenario, Alice and Bob share a physical system and wish to determine whether or not the quantum state they use to describe it is entangled. For this task, one can use the device-dependent approach, which means that both Alice and Bob are in control of their physical systems and experimental apparatuses – they know which measurements they can perform, which kinds of noise and errors are relevant to their setups, what is the efficiency of their detectors, and so on. They could also, for example, perform full state tomography [6, 7] and reconstruct the density matrix of the quantum state associated to their systems. With the knowledge of the quantum state that describes their system, they can check whether or not it is entangled.

On the other hand, in the typical nonlocality scenario, the involved parties do not necessarily have the ability to describe their physical systems or apparatuses. For this reason, they will treat their experiments as black boxes and use the device-independent approach. But since in practice we only deal with physical systems, (noisy) quantum states and (flawed) quantum measurements, why think of black boxes? Because this means that the only information Alice and Bob will be able to extract from their experiment is the probability distributions of getting the possible outcomes for each choice of input. Not being able to characterize their setups, their best choice is to use the device-independent approach and deal only with probability distributions that carry the accessible information of the experiment. The reader will notice that even when we use the device-independent approach we will talk about quantum measurements on quantum states, but our central object will be the probability distributions related to them, and nothing else. To certify the correlations between their systems, Alice and Bob will check whether or not their joint probability distributions are nonlocal.

At last, on the steering scenario, Bob is considered to be able to completely characterize his system, as he can perform full state tomography, while Alice can only analyze and communicate raw statistical data. This is the **semi-device-independent** approach and it is the one on which this dissertation will be focused. It is assumed that Bob has a physical system described by a known quantum state, that he has full control over his experimental setup, and that his laboratory is closed so that neither Alice nor some other third party is able to interact with his systems. His approach will be device-dependent.

Alice's laboratory, on the other hand, is untrusted and she is considered to be dealing with a black box, so her approach will be device-independent. This scenario arises naturally on quantum cryptography [8], if one of the parties may be trusted but the other may not, or when there is the need to certify that a quantum resource is available but there are no means to characterize one of the parties. When choosing this approach to study quantum steering, it is common to call Alice the *steering party* and Bob the *steered party*.

1.2 Information theoretic task

In order to gain some intuition about steering, let us begin by describing it in an operational way, with a steering test, or as it is also called, a semi-device-independent entanglement certification. Say Alice prepares many copies of a bipartite physical system and sends one part of each to Bob, who wants to make tests and find out what is the correlation in the quantum state that describes their physical system. He wishes to do that without having to rely on any information given by Alice but having full trust on his apparatus.

After he received his systems, he will start by asking her to perform measurements on a specific basis of his choosing and communicate her results on each round. According to her results, he can separate his copies in groups and for each choice of measurement and possible outcome she reports, Bob can perform tomography and reconstruct the quantum state that is associated to each input and output obtained by Alice.

Let us say Alice prepared a separable quantum state but she is trying to trick Bob into thinking it is entangled. Let their state be, for example,

$$\rho_{AB} = \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}, \tag{1.1}$$

and suppose Bob asks Alice to perform measurements in the Z basis. She will classically communicate to Bob the results she got each time she performed a measurement and Bob will separate his systems accordingly and perform tomography. He finds out that whenever Alice got the output 0 his conditioned state is $|0\rangle$, and likewise for the output 1. So far, Bob cannot tell whether or not Alice is lying, since there is correlation between their systems. But we know that this correlation is classical! They could have gotten the same result if, for example, Alice had sent him one face of a coin and kept the other. Every time Alice has heads, Bob has tails, and vice versa.

It is important to understand that steering is not only remotely conditioning a quantum state - the resulting correlations must be nonlocal, as we will see in a bit. Bob is aware of this, and will continue the test, now asking Alice to perform measurements in the X basis. Since the state is separable, independently of the outcomes obtained and communicated by Alice, when Bob performs tomography he will see that his systems are in a maximally mixed state. That is, he will find out his state is proportional to the identity, with equal probabilities of getting $|+\rangle$ or $|-\rangle$ *regardless* of Alice's outcomes. Now Bob knows Alice is lying and that their systems are not correlated in the way that she had claimed!

But suppose Alice did send him part of a maximally entangled state, *e.g.*,

$$\rho_{AB} = |\Psi^-\rangle\langle\Psi^-|, \quad \text{where } |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}. \tag{1.2}$$

Now, every time Alice gets the outcomes 0, 1, $+$ or $-$ for her measurements, Bob's conditioned states, that is, the states he obtains by performing tomography on different groups of his subsystem that were separated according to Alice's results, are $|1\rangle$, $|0\rangle$, $|-\rangle$ or $|+\rangle$, respectively. He is convinced that their state is entangled, as Alice was able to *steer* his subsystems and Bob was able to verify that there exists nonlocal correlations between their systems.

However, one could ask "what if Alice lies about her results?", since after all we are starting from the premisse that Alice is untrusted. The whole point is that, by faking her statistics, Alice could never simulate this sort of nonlocal correlations and trick Bob. The only way of convincing him that the quantum state that describes their systems is entangled is by communicating results that come from measurements on a part of a steerable state [3]. If her apparatus fails, if there is noise on her setup, or if she fabricates the results, Bob will think their systems are only classically correlated, even if that is not in fact the case. That is the best Alice is able to do if she lies about her statistics. This test will never yield a "false positive", *i.e.*, yield nonlocal correlations when they do not exist, and this is why it is useful in certifying correlations in a semi-device-independent setting [9, 10].

1.3 Objects of interest

Now that we have gained some perspective on our approach to the study of quantum correlations, let us put some concepts on firmer ground. On each of the following subsections we define some central objects involved in the definition of entanglement, nonlocality and steering. Throughout this work we focus on bipartite scenarios and finite dimensions, unless explicitly stated otherwise.

1.3.1 Quantum states

When dealing with quantum mechanics, one of the first mathematical objects in which one could be interested is the quantum state, defined as follows:

Definition 1.1 (Quantum states). *Let $\mathcal{H}^d$ be a d-dimensional Hilbert space*[1]*. A quantum state $\rho : \mathcal{H}^d \to \mathcal{H}^d$ is a positive semidefinite*[2] *linear operator of unit trace acting on $\mathcal{H}^d$, i.e.,*

(i) $\rho \geq 0$,

(ii) $\mathrm{Tr}\,(\rho) = 1$.

Quantum states are objects that carry the accessible information in a physical system. They are also the main objects of interest in the study of entanglement. As we will see, entanglement (or separability) is a property of quantum states. We denote the set of quantum states by $\mathcal{D}(\mathcal{H})$.

To describe a physical system shared by two parties, we use a bipartite quantum state, or global state.

Definition 1.2 (Bipartite states). *Given two single systems of parties A and B of dimension m and n, respectively, acting on Hilbert spaces $\mathcal{H}_A^m$ and $\mathcal{H}_B^n$, the global bipartite state of this composite system $\rho_{AB} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$ is a quantum state acting on the associated Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A^m \otimes \mathcal{H}_B^n$ of dimension mn.*

[1] For the purpose of this text, a Hilbert space $\mathcal{H}^d$ is a d-dimensional complex vector space equipped with inner product.

[2] An operator $\rho : \mathcal{H}^d \to \mathcal{H}^d$ is positive semidefinite if it satisfies $\langle v|\rho|v\rangle \geq 0 \;\forall\, v \in \mathcal{H}^d$.

The quantum state that describes the physical system of one of the involved parties that share the bipartite state is the reduced state.

Definition 1.3 (Reduced states)**.** *Given a bipartite state ρ_{AB} the reduced state of one of the parties is given by the partial trace of the global state over the other party:*

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}), \tag{1.3}$$

and analogously for ρ_B, where $\mathrm{Tr}_{B(A)}$ denotes the partial trace[3] *over the subspace $\mathcal{L}(\mathcal{H}_{B(A)})$.*

1.3.2 Quantum measurements

To extract information from physical systems, it is necessary to interact with them by means of a quantum measurement, that is represented by a positive-operator valued measure (POVM), defined in the following:

Definition 1.4 (POVMs)**.** *A POVM $\{M_a\}$, with $a \in \{1, \ldots, O\}$, is a set of O positive semidefinite operators $M_a : \mathcal{H}^d \to \mathcal{H}^d$, called POVM elements, that act on a d-dimensional Hilbert space and sum to identity, i.e.,*

(i) $M_a \geq 0 \quad \forall\, a$,

(ii) $\sum_a M_a = \mathbb{1}$.

An interesting particular case of POVM is the projective measurement.

Definition 1.5 (Projective measurements)**.** *A projective measurement $\{\Pi_a\}$, with $a \in \{1, \ldots, O\}$, is an O-outcome POVM, with $O \leq d$, that satisfies the additional condition that all of its measurement elements are projectors onto subspaces of $\mathcal{H}^d$, i.e.,*

$$\Pi_a^\dagger \Pi_a = \Pi_a. \tag{1.4}$$

[3]Partial trace is a linear map $\mathrm{Tr}_B : \mathcal{L}(\mathcal{H}_{AB}) \to \mathcal{L}(\mathcal{H}_A)$ defined by,

$$\mathrm{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \mathrm{Tr}(|b_1\rangle\langle b_2|),$$

and extended by linearity.

Some POVMs that can be performed on a bipartite quantum state are defined as the tensor product of other two POVMs that act locally on the subsystems that compose the bipartite state. These are the bipartite product POVMs.

Definition 1.6 (Bipartite product POVM)**.** *Let $\{M_a\}$, with $a \in \{1, \ldots, O_A\}$, be an O_A-outcome POVM that acts on a m-dimensional Hilbert space $\mathcal{H}_A^m$ and let $\{N_b\}$, with $b \in \{1, \ldots, O_B\}$, be a O_B-outcome POVM that acts on a n-dimensional Hilbert space $\mathcal{H}_B^n$. Each of these POVMs, called local measurements, is performed locally on one of the parts of a bipartite quantum state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A^m \otimes \mathcal{H}_B^n)$. The resulting POVM that acts on the bipartite state, called product POVM, is given by $\{M_a \otimes N_b\}$.*

Not all POVMs that act on composite quantum systems can be written as the tensor product of local POVMs [11]. If quantum states of composite systems exhibit a rich structure, composite POVMs are even richer, since they involve sets of operators, instead of just one operator. However, due to their direct physical interpretation and viable experimental implementation, product POVMs are often studied.

A particular case of bipartite product POVM that will be often used in this dissertation is the POVM whose elements are the tensor product between one local POVM $\{M_a\}$ performed on one part of a bipartite quantum state while the other part performs the trivial one-outcome POVM $\{\mathbb{1}\}$. The resulting product POVM that acts on the bipartite state is $\{M_a \otimes \mathbb{1}\}$. It can be interpreted as one measurement being performed on one subsystem while the other subsystem is left unmeasured.

1.3.3 The Born rule

The probability of obtaining a certain outcome when performing a certain quantum measurement on a given quantum state is given by the Born rule [12].

Definition 1.7 (Born rule)**.** *Let $\{M_a\}$, with $a \in \{1, \ldots, O\}$, be an O-outcome quantum measurement, and ρ be a quantum state. Then, the probability of obtaining the outcome a when performing the measurement $\{M_a\}$ on the state ρ is given by*

$$p(a) = \mathrm{Tr}(M_a\, \rho). \tag{1.5}$$

From a general probability theory point of view, the Born rule can be considered the definition of quantum mechanics itself [13]. Probability distributions that can be obtained via definition 1.7 can be called quantum. They are the central objects in the study of nonlocality with a device-independent approach. In the remaining of this chapter we will see that (non)locality is a property of probability distributions, including the ones that describe quantum physical systems.

A particular case of the Born rule is when the quantum state in eq. (1.5) is a bipartite state and the quantum measurements being performed on it are product POVMs, composed of local measurements acting individually on each subsystem. Then, eq. (1.5) can be rewritten as

$$p(ab|xy) = \mathrm{Tr}(M_{a|x} \otimes M_{b|y}\, \rho_{AB}) \tag{1.6}$$

where ρ_{AB} is a bipartite quantum state, $\{M_{a|x}\}$, with $x \in \{1, \dots, I_A\}$ and $a \in \{1, \dots, O_A\}$, is a set of I_A local measurements on part A with O_A outcomes each and $\{M_{b|y}\}$, with $x \in \{1, \dots, I_B\}$ and $a \in \{1, \dots, O_B\}$, is a set of I_B local measurements on part B with O_B outcomes each. $p(ab|xy)$ is the probability of jointly obtaining outcomes a and b when performing measurements x and y.

A set of probability distributions for a fixed number of inputs on each party and fixed number of possible outcomes for each measurement is called **box**. In our case, we will be dealing with **bipartite boxes**, sets of probability distributions involving two parties, where Alice(Bob) is allowed to choose between $I_{A(B)}$ inputs with $O_{A(B)}$ possible outcomes each.

A characteristic of all bipartite boxes that can be obtained via Born rule is the *nonsignaling* property. Assuming that no communication is allowed between the parties, the nonsignaling property is the fact that the choice of input of one party cannot influence the probability of the other party getting a certain outcome for any of its choices of input. Formally, it means that

$$p_A(a|x) = \sum_b p(ab|xy) \;\; \forall\, y, \qquad p_B(b|y) = \sum_a p(ab|xy) \;\; \forall\, x. \tag{1.7}$$

1.3.4 Assemblages

When studying steering, the mathematical object of interest is the assemblage. It carries both the information of a device-independent approach to Alice, a probability distribution, and the information of a device-dependent approach to Bob, a conditioned quantum state.

Consider that on her black box, Alice can choose among $x \in \{1, \ldots, I_A\}$ inputs and will get one of $a \in \{1, \ldots, O_A\}$ outcomes with probability $p_A(a|x)$. On each round she will classically communicate her choice of input and obtained outcome to Bob. Bob will use the information provided by Alice to separate his systems and perform state tomography. For each pair (a, x) that Alice communicates, he will obtain the single-party quantum state $\rho_{a|x}$. Then, assemblage held by Bob is defined as follows.

Definition 1.8 (Assemblages). *Given an ordered pair $(p_A(a|x), \rho_{a|x})$ composed of an element of a probability distribution obtained by Alice and a single-party quantum state held by Bob, the resulting assemblage $\{\sigma_{a|x}\}_{a,x}$ is a set of subnormalized positive semidefinite operators $\sigma_{a|x} : \mathcal{H}_B \to \mathcal{H}_B$ given by*

$$\sigma_{a|x} = p_A(a|x)\rho_{a|x}. \tag{1.8}$$

If the assemblage is a result of Alice performing local quantum measurements on her part of a global quantum state shared with Bob, then it is possible to write

$$\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}_B \, \rho_{AB}), \tag{1.9}$$

where ρ_{AB} is a bipartite global state and $\{M_{a|x}\}_{a,x}$, with $x \in \{1, \ldots, I_A\}$, is a set of I_A measurements with $a \in \{1, \ldots, O_A\}$ outcomes each.

For the case of bipartite scenarios, all assemblages can be written as resulting from local measurements on one part of a bipartite quantum state [14–16]. For this reason, equations (1.8) and (1.9) will be used as equivalent definitions of an assemblage.

Notice that:

$$\mathrm{Tr}(\sigma_{a|x}) = \mathrm{Tr}\left(p_A(a|x)\, \rho_{a|x}\right) = p_A(a|x), \tag{1.10}$$

since $\rho_{a|x}$ is a quantum state and thus has trace 1. Since $p_A(a|x) \in [0, 1]$ and $\sum_a p_A(a|x) = 1$, the assemblage element is subnormalized (has trace less than one) unless a measurement with a single outcome is performed by Alice. Since both $p_A(a|x) \geq 0$ and $\rho_{a|x} \geq 0$, the assemblage elements are positive semidefinite operators. Hence, each assemblage element is a subnormalized quantum state.

It is important to notice that every assemblage satisfies this condition:

$$\sum_a \sigma_{a|x} = \sum_a p_A(a|x)\rho_{a|x} = \rho_B \quad \forall x. \tag{1.11}$$

This can be interpreted as a nonsignaling condition, just as for bipartite boxes in eq. (1.7). The convex sum of all of Bob's conditioned states for a fixed measurement performed by Alice on her subsystem must return the reduced state of Bob. This means that by performing a measurement, Alice cannot affect the probability distributions of Bob's subsystem unless she classically communicates her results to Bob.

1.4 Entanglement

Now we can begin to establish exactly what are each of these three important quantum correlations – entanglement, nonlocality, and steering – with precise mathematical definitions. The goal of this section is to define entanglement so that its role in steering and nonlocality can be better understood. We do not intend to explore or review the theory of entanglement. For those matters, the reader is referred to the review in reference [17] and the thesis in reference [18].

1.4.1 Separability

Instead of defining entanglement, let us start by defining the absence of entanglement: separability.

Definition 1.9 (Separability)**.** *A finite dimensional bipartite state ρ_{AB} is separable if it can be written in the form*

$$\rho_{AB} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i \tag{1.12}$$

where $p_i \geq 0$, $\sum_i p_i = 1$ is a probability distribution and ρ_A and ρ_B are single-party quantum states.

The set of all separable states is denoted SEP. Simply enough:

Definition 1.10 (Entanglement)**.** *A finite dimensional bipartite quantum state ρ_{AB} is entangled if it cannot be written in the form of eq.* (1.12)*.*

Even when the global state of a system is known, checking whether or not the state is entangled is usually not a simple task. For the cases of bipartite states acting on $\mathcal{H}^2 \otimes \mathcal{H}^2$ or $\mathcal{H}^2 \otimes \mathcal{H}^3$, that is, two-qubit states and qubit-qutrit states, the Peres-Horodecki criterion [19, 20], based on the positivity of the partial transpose of a quantum state, is decisive to determine whether or not a quantum state is entangled. However, even for a state shared by only two parties with finite low dimension, like the case of qutrit-qutrit states, guaranteeing that there does not exist a separable decomposition, given by eq. (1.12), for the global state can be a difficult problem [21]. Instead, one can check for entanglement witnesses.

1.4.2 Entanglement witnesses

Entanglement witnesses are Hermitian operators that can be physically interpreted as observables whose expectation value on a certain quantum state can be used as a sufficient criterion to certify that the state is entangled [20, 22].

Definition 1.11 (Entanglement witness)**.** *Given a Hermitian operator W, it constitutes an entanglement witness if for some $\rho \in \mathcal{D}(\mathcal{H})$,*

$$\mathrm{Tr}(W\rho) < 0,$$

while for every separable state $\sigma \in SEP$,

$$\mathrm{Tr}(W\sigma) \geq 0.$$

Geometrically, these inequalities define a hyperplane[4] in the set of quantum states that separates an entangled state from the subset of separable states.

Let us look at the geometric representation in fig. 1.1: given a Hermitian operator W, the hyperplane defined by it is the region of all states for which W has zero expectation value. All separable states have non-negative expectation value for W and *some* entangled

[4] Given a d-dimensional vector space, a hyperplane is a $(d-1)$-dimensional affine subspace of this vector space, when well defined, that is characterized by a linear equation. It is the generalization of the notion of plane in a three-dimensional Euclidian space to arbitrary dimensions in any vector space.

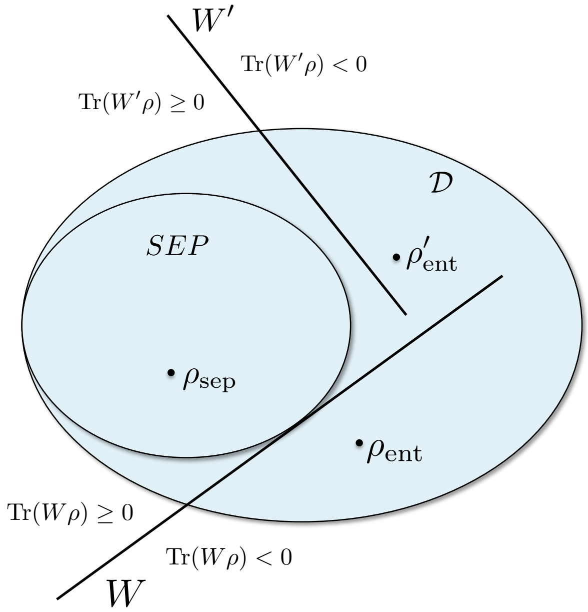

Figure 1.1: The entanglement witness W is able to detect the entangled state ρ_{ent} but does not detect the entangled state ρ'_{ent}, which is, in turn, detected by the witness W'. All separable states ρ_{sep} satisfy all entanglement witnesses (have non-negative expectation value for all W).

states have negative expectation value for W. These are the entangled states that the witness W is able to detect. Other entangled states have non-negative expectation value for W and are not be detected by this witness. However, they can be detected by some other witness W' [20].

It is evident that there does not exist only one entanglement witness - any operator satisfying definition 1.11 is a witness for some entangled state. For this reason, a fixed witness is only a sufficient, and not necessary, criterion for entanglement. This is a consequence of the fact that the set of separable states SEP is closed and convex. Still, we can use this concept to redefine entanglement:

Definition 1.12 (Entanglement). *A quantum state ρ is entangled if, for some entanglement witness W, it holds that*

$$\mathrm{Tr}(W\rho) < 0.$$

1.5 Nonlocality

Once more, the purpose of this section is to define nonlocality in order to allow a comparison between this concept, entanglement, and steering in the remaining of this dissertation. For a deeper study of nonlocality in quantum mechanics we recommend references [9, 23, 24]. Before defining nonlocality, there is a handful of important concepts one should have in mind that will be useful not only here, but when talking about steering as well. The first of them is the convex set.

Definition 1.13 (Convex set)**.** *Let $\mathcal{C}$ be a set in some real vector space $\mathcal{V}$. The set $\mathcal{C}$ is convex if, for every two elements $c_1, c_2 \in \mathcal{C}$, their convex combination is also an element of $\mathcal{C}$, that is*

$$c = tc_1 + (1-t)c_2 \in \mathcal{C}, \quad \forall\, t \in [0,1]. \tag{1.13}$$

Geometrically, this means that given two points in $\mathcal{C}$ in an Euclidian space, by connecting them with a straight line segment, all points in this segment also belong to $\mathcal{C}$. The set of separable states SEP is a convex set [17]. As we will see, so is the set of local boxes [9] and the set of unsteerable assemblages [25, 26].

There are some points that belong to a convex set $\mathcal{C}$ that cannot be described as a convex combination of other points in $\mathcal{C}$ but, in turn, all other points can be written as a convex combination of them. These are called the **pure points** or **extremal points** of $\mathcal{C}$. For example, picture a two-dimensional square. All points strictly inside the square can be written as a convex combination of the points in the edges; all points in the edges can be written as a convex combination of the four points in the vertices. However, the vertices cannot be written as a convex combination of any other[5] points on the square, but they are still elements of the "set square". They are its extremal points. Another example is a two-dimensional circle. Any point on the boundary of the circle cannot be written as a convex combination of points inside the circle. Hence, all points that belong to the boundary of the "set circle" are extremal. Notice that, contrary to the square, the circle has an infinite number of extremal points.

[5] Notice that extremal points can only be written as a trivial convex combination of themselves with $t = 1$.

Given any set $\mathcal{A} \in \mathcal{V}$, convex or not, the **convex hull** of $\mathcal{A}$ is the smallest convex set that contains all elements of $\mathcal{A}$. Convex hulls can be used to define an important type of convex set: the convex polytope.

Definition 1.14 (Convex polytope)**.** *A convex polytope is the convex hull of a finite number of points.*

The "edges" of the convex polytope are called **facets**, and they correspond to hyperplanes in the vector space that contains the convex set. A convex polytope, or as we call it, a polytope, has a finite number of facets and extremal points. From the previous example, the set square is a polytope, although the set circle is not. A polytope can be fully characterized by its extremal points or equivalently by its facets.

1.5.1 Local hidden variable models

Now we are ready to define nonlocality, or else, locality [27].

Definition 1.15 (Locality)**.** *A bipartite box is local if there exists a local hidden variable (LHV) model that describes every probability distribution in it. That is, there exists a local hidden variable Λ such that, for all inputs x and y and outputs a and b, the following holds:*

$$p(ab|xy) = \sum_{\lambda} \pi(\lambda)\, p_A(a|x,\lambda)\, p_B(b|y,\lambda) \quad \lambda \in \Lambda, \tag{1.14}$$

where λ are the values that Λ can assume with probability $\pi(\lambda)$, $p_A(a|x,\lambda)$ is the probability of part A obtaining the outcome a conditioned to the inputs x and to the value λ of the local hidden variable, and equivalently for $p_B(b|y,\lambda)$.

To the reader that is not familiar with the theory of nonlocality, this definition can seem difficult to interpret. But the idea behind a **local hidden variable** description is that, given a black box for each party, the correlation between the events of getting a and b when choosing x and y can be explained by a common cause, the variable Λ, that is *independent* from the choices of x and y. The LHV model itself consists of a probability distribution π over the possible values of the variable Λ and the so-called **response functions** p_A and p_B, that attribute a probability for each outcome that Alice and Bob, respectively, can obtain for a given input depending on the value assumed by Λ. The local hidden variable is also called **shared randomness** [9]. To say that a box admits an LHV model means that there exist Λ, π, p_A, and p_B that can mimic all probability distributions of the box, according to eq. (1.14).

One can look at $p(ab|xy)$ as the joint choice of x and y influencing the probability of obtaining the joint result a and b. However, for local boxes, the variable Λ can be seen as influencing the probability distribution for Alice and for Bob *separately*. With the knowledge of Λ we can reveal that the apparent correlation between Alice and Bob is actually a correlation between each part and a common local hidden variable, instead of between each other. The parts only seem correlated because of our lack of knowledge of this extra variable.

The set of local boxes, denoted $\mathcal{L}$, is a convex polytope [24, 28, 29], and therefore can be characterized by a finite number of extremal points or facets.

For a further reading on LHV models we recommend the first chapter of reference [30], the third chapter of reference [31], and the review in reference [32].

Just as for a separable quantum state there exists a separable decomposition (eq. (1.12)), for local boxes there exits an LHV model. And just as entangled quantum states can be detected by entanglement witnesses, nonlocal boxes can be witnessed as well, by means of Bell inequalities.

1.5.2 Bell inequalities

Bell inequalities were first proposed by John Bell in the seminal paper [33], in response to the apparent EPR [2] paradox, as a mathematical formalization of the concept of quantum nonlocality.

For our definition of Bell inequalities, we use the concept of **measurement events** [34]. Measurement events e are the occurrence of result a for measurement x (in this case, $e = a|x$), or the occurrence of result a and b for measurement x and y ($e = ab|xy$), and so on.

Think of a linear functional over elements of probability distributions on measurement events that returns a real number. If this function acts on a local box, *i.e.*, a box with probability distributions that can be described by a local model (eq. (1.14)), then there is a maximum value that this linear function can achieve [9]. This value is called the local bound and, along with the coefficients of the linear function, constitutes a Bell inequality.

Definition 1.16 (Bell inequality)**.** *Given a set of coefficients c_e and a local bound S^{loc}, a Bell inequality is defined as*

$$S = \sum_e c_e \, p(e) \leq S^{loc} \tag{1.15}$$

where e represents measurement events, $p(e)$ are the probabilities of occurrence of these measurement events, and S^{loc} is the maximum value that the function S can achieve for a local box, namely,

$$S^{loc} = \max_{p \in \mathcal{L}} \sum_e c_e \, p(e). \tag{1.16}$$

The idea behind these inequalities does not really have anything to do with quantum mechanics: they are more of a statement about our classical intuition. So, if you believe nature is local and all correlations can be explained by local hidden variable models, then the local bound should never be exceeded by probability distributions that come from measurements on physical systems. But as it turns out, nature, just like quantum theory, does not obey locality[6] and experiments have demonstrated that Bell inequalities can be violated [35–38]. These violations are predicted by quantum theory and the boxes that violate Bell inequalities are called **nonlocal**, in the sense of Bell[7].

Definition 1.17 (Nonlocality)**.** *A box is nonlocal if it does not admit a local hidden variable description. Equivalently, a box is nonlocal if it violates a Bell inequality.*

Nonlocal boxes cannot be described by LHV models, which can be difficult to assimilate since we have made such little assumptions about the nature of the local hidden variable. For this reason, nonlocality has been thought of as one of the most intriguing aspects of nature and one of the most non-intuitive aspects of quantum theory [9].

[6] By locality here we mean the assumption of local realism in which the derivation of Bell's theorem in reference [33] is embedded.

[7] It is important to highlight that nonlocality, in the sense of Bell, does not refer to nonlocal interactions, action or communication at a distance, superluminal signaling, nor other meanings that can be attributed to the term nonlocal in other contexts. We refer exclusively to the definitions proposed in this text that follow from the work of Bell [33].

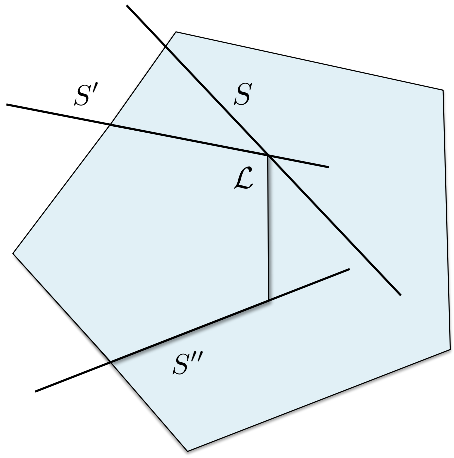

Figure 1.2: Bell inequalities define hyperplanes in the vector space of probability distributions. The tight Bell inequalities that define the hyperplanes $S' = S'^{\text{loc}}$ and $S'' = S''^{\text{loc}}$ are facets of the polytope of local boxes $\mathcal{L}$. The hyperplane $S = S^{\text{loc}}$ is not a facet of the polytope but is also a witness of nonlocality.

Much in the same way as entanglement witnesses, Bell inequalities define hyperplanes in the vector space of probability distributions. But unlike entanglement, the set of local boxes is a polytope and can be characterized, on a fixed scenario, by a finite number of facets, as geometrically represented in fig. 1.2. This means that there exists a finite number of Bell inequalities that are sufficient to characterize all the facets of the local polytope, which is the set of local boxes, while it is necessary an infinite number of entanglement witnesses to characterize the set of separable states. This does not mean the problem is simple - for a high number of parameters, the task of characterizing these facets with linear inequalities quickly becomes difficult [29].

However, if one finds all Bell inequalities that characterize the local polytope, there is a way to test whether or not a box is local without having to explicitly construct an LHV model. If a box satisfies *all* Bell inequalities in this scenario, we can assure it is local. If it violates at least *one* Bell inequality, it is nonlocal.

There are also Bell inequalities that define hyperplanes that do not match the facets of the polytope, even though they touch the polytope at some point, like the hyperplane $S = S^{\text{loc}}$ in fig. 1.2. These inequalities are commonly called *not tight*.

What if instead of probability distributions we want to talk about nonlocality of quantum states? Let us fix a scenario where each party is allowed to perform a number of quantum measurements with some number of possible outcomes each. To affirm that a given quantum state is local in this scenario, it is sufficient to build a local model for the probability distributions that arise from any possible set of quantum measurements of each party acting on the given quantum state. Equivalently, it could be shown that, for any set of measurements, the resulting probability distributions satisfy all tight[8] Bell inequalities in this scenario. To show that a quantum state is nonlocal, it is sufficient to find one set of quantum measurements for each party that allows the resultant probability distributions to violate one (tight or not) Bell inequality. We denote the set of local quantum states as LOC.

For further reading on Bell inequalities we recommend references [9, 39].

1.6 Steering

As mentioned in the introduction, the phenomenon of quantum steering was first described by Einstein, Podolsky, and Rosen in 1935 [2] as some sort of "spooky action at a distance". A little later on, on 1935, Schrödinger [1, 14] gave the name **steering** to the ability of one party to affect the state of the other by her choice of measurement basis, in an attempt to formalize the phenomenon described by EPR. It was not until 2007 that Wiseman, Jones, and Doherty [3] formalized this concept, facilitating the understanding of the phenomenon behind the apparent EPR paradox.

After introducing an information theoretic task in section 1.2 that is commonly used to illustrate steering and discussing about some relevant aspects of entanglement and nonlocality, we are ready to present the formalization of steering-related concepts.

We will continue to work on a bipartite scenario with finite dimension but from now on there is one more important characteristic: Alice wants to steer Bob. This is important because, unlike entanglement and nonlocality, steering is *directional* [40]. By stating that Alice steers Bob, we mean that Alice will be treated as the untrusted party. She is the one with a black box and access only to statistical data and will be treated with a device-independent approach. Meanwhile, Bob's setup is trusted and "quantum", *i.e.*, he

[8] Tight Bell inequalities are facets of the local polytope.

is considered to be dealing with physical systems described by known quantum states and performing known quantum measurements. He will be treated with a device-dependent approach.

1.6.1 Local hidden state models

Steerability is a priori a characteristic of assemblages, and it is the name we give to the ability to demonstrate steering. The assemblages that have this ability are called **steerable**. When an assemblage does not demonstrate steering it is called **unsteerable**.

Analogously to separable states and local boxes, we can build unsteerable models for unsteerable assemblages. In this case, they are called local hidden state models [3].

Definition 1.18 (Unsteerability). *An assemblage $\{\sigma_{a|x}\}$ is unsteerable if it can be described by a local hidden state (LHS) model. That is, if there exists Λ such that for all elements of the assemblage $\sigma_{a|x}$ it holds that*

$$\sigma_{a|x} = \sum_{\lambda} \pi(\lambda)\, p_A(a|x,\lambda)\, \rho_\lambda, \quad \lambda \in \Lambda, \tag{1.17}$$

where λ are the values that the variable Λ can assume with probability $\pi(\lambda)$, $p_A(a|x,\lambda)$ is the probability of Alice getting result a, conditioned to her choice of input x and the value Λ has assumed, and ρ_λ is a local hidden state for Bob.

An assemblage is unsteerable if it admits an LHS model. An LHS model consists of a variable Λ that assumes value λ according to the probability distribution π, a response function p_A for Alice, that provides the probability of obtaining outcome a for choice of input x given that Λ assumed a certain value, and finally a collection $\{\rho_\lambda\}_\lambda$ of local hidden states for Bob. A local hidden state is a quantum state for Bob's subsystem that is determined by the value of the variable Λ. These quantum states, along with Alice's response function and probability distribution over the values of Λ, can classically mimic an unsteerable assemblage.

A generic assemblage is a collection of products of probability distributions $p_A(a|x)$ (over Alice's outcomes conditioned to her choices of inputs) and quantum states $\{\rho_{a|x}\}$ for Bob (conditioned by Alice's inputs and outcomes). In addition, unsteerable assemblages can be interpreted in another way. Instead of Alice's choice of input and obtained outcome being responsible for determining Bob's state, it is possible to consider a variable Λ, independent from Alice's inputs, that is common cause to both Alice's outcome and Bob's conditioned state. Whenever Λ assumes a certain value λ, with probability $\pi(\lambda)$ independent of Alice's choice of input, Alice's probability of obtaining result a to the input x is dependent on the value of λ. On the other side, Bob receives a quantum state ρ_λ from a collection of quantum states $\{\rho_\lambda\}_{\lambda\in\Lambda}$ determined by the value λ, independently of Alice's inputs and outcomes.

Following the reasoning in reference [26], let us rewrite Alice's probability distributions $p_A(a|x,\lambda)$. For a fixed number of inputs I_A and possible outcomes O_A, a probability distribution can be written as a convex combination of **deterministic** probability distributions. Deterministic probability distributions are the ones whose elements can only assume value 0 or 1, *i.e.*, given a choice of input, it is known with certainty what the outcome will be ($p = 1$), and that other outcomes are not possible ($p = 0$). Thus, we can rewrite

$$p_A(a|x,\lambda) = \sum_{\lambda'} \mu(\lambda'|\lambda)\, D(a|x,\lambda'), \tag{1.18}$$

where $D(a|x,\lambda')$ are deterministic probability distributions and $\mu(\lambda'|\lambda)$ is the weight of the convex sum, given λ. Substituting in eq. (1.17),

$$\sigma_{a|x} = \sum_{\lambda} \pi(\lambda) \sum_{\lambda'} \mu(\lambda'|\lambda)\, D(a|x,\lambda')\, \rho_\lambda \tag{1.19a}$$

$$= \sum_{\lambda'} D(a|x,\lambda') \sum_{\lambda} \pi(\lambda)\, \mu(\lambda'|\lambda)\, \rho_\lambda \tag{1.19b}$$

$$= \sum_{\lambda'} D(a|x,\lambda')\, \sigma_{\lambda'}, \tag{1.19c}$$

where $\sigma_{\lambda'} = \sum_\lambda \pi(\lambda)\, \mu(\lambda'|\lambda)\, \rho_\lambda$. Following from the definition of $\sigma_{\lambda'}$ we have:

(i) For each λ' and λ, $\pi(\lambda) \geq 0$ since they are elements of probability distributions, $\mu(\lambda'|\lambda) \geq 0$ since they are weights of the convex sum, and $\rho_\lambda \geq 0$ since it is a quantum state. It follows that $\sigma_{\lambda'} \geq 0\ \forall\, \lambda' \in \Lambda'$.

(ii) Since $\mathrm{Tr}(\rho_\lambda) = 1$, it follows $\mathrm{Tr}(\sigma_{\lambda'}) = \sum_\lambda \pi(\lambda)\, \mu(\lambda'|\lambda) = q(\lambda') \in [0,1]$.

Therefore, $\sigma_{\lambda'}$ is a subnormalized quantum state and consequently $\{\sigma_{\lambda'}\}_{\lambda'}$ fits the definition of an assemblage. Relabeling $\lambda' \to \lambda$, eq. (1.17) is equivalent to

$$\sigma_{a|x} = \sum_{\lambda} D(a|x,\lambda)\, \sigma_{\lambda}, \quad \lambda \in \Lambda. \tag{1.20}$$

We can interpret this equivalent definition of an LHS model: an unsteerable assemblage is one that, for a local hidden variable Λ, can be simulated by a deterministic *strategy* for Alice, that announces deterministically the output a every time the LHV assumes value λ and she chooses the input x. Bob, also deterministically, holds the local hidden (subnormalized) state σ_{λ}, that is independent from a and x.

However, in the same way that there are quantum states that do not admit a separable decomposition and probability distribution boxes that do not admit LHV models, there are assemblages that do not admit an LHS model. These assemblages are called steerable and can be used to demonstrate steering.

Definition 1.19 (Steerability)**.** *An assemblage is steerable if it cannot be written in the form of eq.* (1.17)*, i.e., if it does not admit an LHS model.*

The set of unsteerable assemblages, denoted by $\mathcal{U}$, is a closed, limited, and convex set, but not a convex polytope [41]. This can make difficult the task of characterizing it. However, analogously to entanglement witnesses and Bell inequalities, we can define *steering inequalities* that can witness steerable assemblages.

1.6.2 Steering inequalities

Steering inequalities were first derived in reference [41] as a sufficient criterion for steerability. They are analogous to Bell inequalities: a linear function, now over elements of the assemblage, that returns a real number that is upper bounded by the maximum value that an unsteerable assemblage can achieve for this linear function.

Definition 1.20 (Steering inequality)**.** *Consider a set of $I_A O_A$ Hermitian operators $\{F_{a|x}\}_{a,x}$ that act on a Hilbert space of same dimension as the assemblage $\{\sigma_{a|x}\}_{a,x}$, and define a linear functional β over the assemblage elements. A steering inequality, defined* *by the operators $F_{a|x}$ and the unsteerable bound β^{uns}, is given by*

$$\beta = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}) \leq \beta^{uns}, \tag{1.21}$$

where β^{uns} is the maximum value that the function β can assume for an unsteerable assemblage,

$$\beta^{uns} = \max_{\{\sigma_{a|x}\}\in\mathcal{U}} \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}). \tag{1.22}$$

These inequalities define hyperplanes in the vector space of the assemblages.

Remember that Bob is able to reconstruct his assemblage by using the information provided by Alice and the result of his state tomography. Since Bob knows the assemblage, given $\{F_{a|x}\}$ and β^{uns}, he is able to test the steering inequality in many ways, one of which is to simply calculate the value of β by hand and check whether or not his assemblage violates the inequality.

Steering inequalities work quite in the same way as Bell inequalities. However, recall that the set of unsteerable assemblages is not a polytope, so there is no finite number of inequalities that characterize it. The violation of a steering inequality guarantees steerability, but to satisfy one does not imply unsteerability.

Crucially, the violation of a steering inequality says something very important about the quantum state that generated that assemblage. Even though it is not possible to fully characterize the global state shared by Alice and Bob, the violation of a steering inequality garanties that the shared state is entangled. A steering test is able to certify this useful property based only on some statistical data from Alice's subsystems and tomography reconstruction of the state of Bob's subsystems, *i.e.*, in a semi-device-independent manner. Alice and Bob might not know exactly what state they share, but by violating a steering inequality (or a Bell inequality) they are assured that their state is entangled.

Steering inequalities can be used to provide a definition of steering that is equivalent to definition 1.19.

Definition 1.21 (Steering)**.** *An assemblage is steerable if it violates a steering inequality.*

At this point, it is interesting to reanalyze Alice's role in the steering test proposed in section 1.2. Her job in this task is to perform local quantum measurements on her subsystems and communicate her results to Bob. Since their systems are correlated, by gaining the information provided by Alice, Bob is able to learn more about his own subsystems and update the description he has of them. Hence, Alice's role is to provide Bob with partial information about their global state that he would not be able to obtain by interacting with only his subsystem. Alice is, in fact, not altering or affecting Bob's state directly. For this reason, steering is not to be confused with some sort of action at a distance, as Alice is only classically providing Bob some additional information.

If one wishes to discuss the steerability of quantum states instead of assemblages, the same reasoning as for nonlocality can be followed. Let us fix a scenario of I_A inputs and O_A possible outcomes. Given a quantum state, if there exist an LHS model for the assemblage that results from every possible measurement set allowed in this scenario on the fixed quantum state, we say this state is unsteerable in this scenario. A quantum state is truly unsteerable if it is unsteerable in every possible scenario, *i.e.*, if for all sets of quantum measurements, all resulting assemblages from the given quantum state are unsteerable. If for one set of measurements on this quantum state a steering inequality is violated, then the state is steerable. We denote the set of unsteerable states as UNS.

For further reading on steering inequalities we suggest references [42, 43].

To summarize the information provided here on entanglement, steering, and nonlocality, the main aspects of each quantum correlation are put in comparison on table 1.1. This table is adapted from the work of Matthew Pusey on reference [25].

1.7 Werner states and the set of quantum states

We have seen that some quantum states are entangled, some are steerable, and some are nonlocal. But is there an intersection among these sets of quantum states?

Yes, there is. More than an intersection among these sets, there is a hierarchy among these correlations. Let us start investigating this by taking a separable state – one that admits a separable decomposition.

$$\rho_{AB}^{\text{sep}} = \sum_{\lambda} p_\lambda \, \rho_\lambda^A \otimes \rho_\lambda^B. \tag{1.23}$$

By making measurements on one part of this quantum state, the resulting assemblage is

	Entanglement	Steering	Nonlocality
Trusted Parties	Alice and Bob	Bob	Neither
Parameters	d_A, d_B	I_A, O_A, d_B	I_A, O_A, I_B, O_B
Object of interest	$\rho_{AB} \in \mathcal{D}(\mathcal{H}^{d_A d_B})$	$\{\sigma_{a\|x}\} \in \mathcal{L}(\mathcal{H}^{d_B})$	$\{p(ab\|xy)\} \in \mathbb{R}$
Positivity	$\rho_{AB} \geq 0$	$\sigma_{a\|x} \geq 0 \ \forall a, x$	$p(ab\|xy) \geq 0 \ \forall a, b, x, y$
Normalization	$\mathrm{Tr}(\rho_{AB}) = 1$	$\sum_a \mathrm{Tr}(\sigma_{a\|x}) = 1 \ \forall x$	$\sum_{a,b} p(ab\|xy) = 1 \ \forall x, y$
No-signaling A $\to$ B	N/A	$\rho_B = \sum_a \sigma_{a\|x} \, \forall x$	$p(b\|y) = \sum_a p(ab\|xy) \, \forall x$
No-signaling B $\to$ A	N/A	N/A	$p(a\|x) = \sum_b p(ab\|xy) \, \forall y$
Certifying absence	Separable decomposition	LHS model	LHV model
Certifying presence	Entanglement witness	Steering inequality	Bell inequality

Table 1.1: Comparison between different aspects of entanglement, steering, and nonlocality.

always be unsteerable, admitting an LHS model:

$$\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, \rho_{AB}^{\mathrm{sep}}) \tag{1.24a}$$
$$= \sum_\lambda p_\lambda \, \mathrm{Tr}(M_{a|x} \rho_\lambda^A) \, \rho_\lambda^B. \tag{1.24b}$$

In this case, the LHS model is given by: Alice's response function $p_A(a|x, \lambda) = \mathrm{Tr}(M_{a|x}\rho_\lambda^A)$ is the probability distribution of the outcome of her measurements on her single system states, Bob's conditioned states $\rho_\lambda = \rho_\lambda^B$ are his collection of single system quantum states and the probability distribution over the values of the local hidden variable $\pi(\lambda) = p_\lambda$ are the weights of the separable decomposition. This is true for any set of measurements performed locally by Alice on a separable state.

Now take an unsteerable state – one that admits LHS models for any set of measurements,

$$\sigma_{a|x}^{\mathrm{uns}} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1} \rho_{AB}^{\mathrm{uns}}) \tag{1.25a}$$
$$= \sum_\lambda \pi(\lambda) p_A(a|x, \lambda) \rho_\lambda. \tag{1.25b}$$

This kind of state can only yield local sets of probability distributions, that always admits an LHV model:

$$p(ab|xy) = \mathrm{Tr}(M_{b|y}\sigma^{\mathrm{uns}}_{a|x}) \tag{1.26a}$$

$$= \sum_{\lambda} \pi(\lambda)\, p_A(a|x,\lambda)\, \mathrm{Tr}(M_{b|y}\rho_\lambda). \tag{1.26b}$$

This LHV model constructed from measurements on an unsteerable assemblage has the same probability distribution over Λ than the LHS model, same response function for Alice and Bob's response function $p_B(b|y,\lambda) = \mathrm{Tr}(M_{b|y}\rho_\lambda)$ is given by the probability of obtaining outcome b when performing measurement y on the collection of single system states $\{\rho_\lambda\}$.

This means that all separable states are unsteerable and all unsteerable states are local, which on its turn means that entanglement is *necessary* for steerability and steerability is *necessary* for nonlocality.

Nevertheless, the converse does not hold. For instance, start with a local box that comes from measurements on generic quantum states:

$$p(ab|xy) = \mathrm{Tr}(M_{a|x} \otimes M_{b|y}\, \rho_{AB}) \tag{1.27a}$$

$$= \sum_{\lambda} \pi(\lambda)\, p_A(a|x,\lambda)\, p_B(a|x,\lambda). \tag{1.27b}$$

Even though the joint probability distribution $\{p(ab|xy)\}$ results from measurements on a quantum state, it is not possible to guarantee that the response functions of Alice and Bob, $p_A(a|x,\lambda)$ and $p_B(a|x,\lambda)$, are equivalent to probability distributions yielded by quantum measurements on single system quantum states! Consequently, it is not possible to guarantee that the global quantum state that generated the joint probability distributions is unsteerable or separable. Indeed, there are local states that are steerable and unsteerable states that are entangled. This means that entanglement is *not sufficient* for steerability and steerability is *not sufficient* for nonlocality. This result has been proven both for projective measurements [3] and for general POVMs [44]. Entanglement, EPR steering, and Bell nonlocality are proven to be genuinely different phenomena[9]. In terms of the set of quantum states, this means that

$$SEP \subset UNS \subset LOC \subset \mathcal{D}. \tag{1.28}$$

[9]At least for mixed states. It has been proved in reference [45] that any pure entangled state of two or more parties is nonlocal and, hence, steerable. But for mixed states, such as some Werner states, this does not hold for the strict sense of steering and nonlocality defined in this text.

Since entanglement is necessary for exhibiting steering and nonlocality, we can use tests of these correlations to certify entanglement in a semi-device-independent and device-dependent way, respectively. Theoretically, the set of entangled states that can be detected via a steering test is larger than via a Bell test, since one could in principle detect local steerable states. However, there is the extra difficulty that one of the parties must be guaranteed to be trusted. A pictorial representation of the set of quantum states and the correlations in them can be found on fig. 1.3.

An interesting tool to study the hierarchy between entanglement, steering, and nonlocality are the Werner states in two dimensions[10] [27], a family of two-qubit quantum states given by

$$\rho_W(\eta) = \eta|\Psi^-\rangle\langle\Psi^-| + (1-\eta)\frac{\mathbb{1}}{4}, \quad \eta \in [0,1], \tag{1.29}$$

where

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{1.30}$$

is the singlet state, a maximally entangled two-qubit state. In this dissertation, 2-dimensional Werner states are simply called Werner states.

If η assumes the value 1, the Werner state $\rho_W(1)$ is equal to the singlet. If it assumes value 0, $\rho_W(0)$ equals the normalized identity. But for other values of η in the range $[0,1]$ we can detect a variety of correlations.

It is known that $\rho_W(\eta)$ is separable for $\eta \leq \frac{1}{3}$ [27]. The Werner states admit LHV models for all general POVMs for $\eta \leq \frac{5}{12}$ [46]. They admit LHS models for all projective measurements for $\eta \leq \frac{1}{2}$ [27] and LHV models for all projective measurements for $\eta \leq 0.6595$ [47]. It is also known that the Werner states violate a Bell inequality, the CHSH inequality [39] for $\eta > \frac{1}{\sqrt{2}}$.

In 2006, Acín, Gisin, and Toner [47] showed that the exact "critical" value for η bellow which the Werner states are local and above which they are nonlocal, is the inverse of the Grothendieck constant [48, 49] of order 3, $K_G(3)$, although the exact value of this constant is not known. The best that is known of it are lower and upper bounds. An

[10] Werner states are d^2-dimensional bipartite quantum states that are invariant when the same unitary transformations is applied locally on each part of the state, therefore being highly symmetrical states. Two-qubit Werner states are equivalent to a convex combination of the singlet state and the maximally mixed state. Higher dimensional Werner states do not necessarily have this form. Yet, there is another family of quantum states, called the isotropic states, that can be defined as a convex combination of the maximally entangled bipartite state in dimension d^2 and the maximally mixed state in dimension d^2.

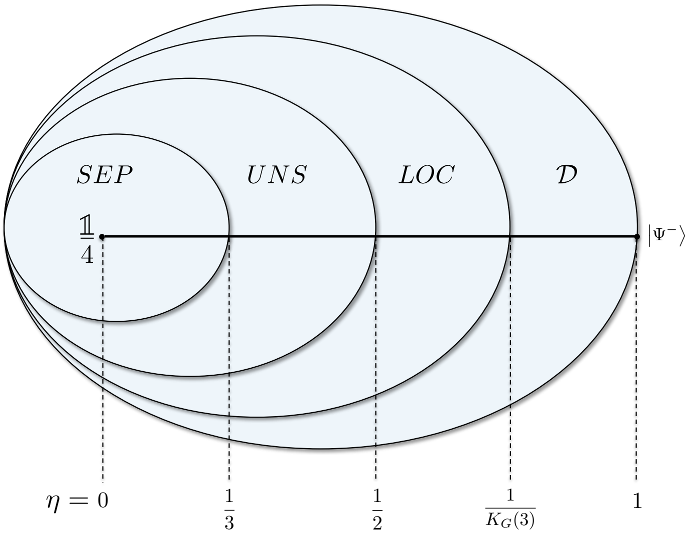

Figure 1.3: Geometric representation of $SEP \subset UNS \subset LOC \subset \mathcal{D}$. The line segment connecting the identity (at the center of the set of separable states) to the singlet (at the boundary of the set of all quantum states) represents all possible values for $\eta \in [0, 1]$, the parameter of the convex combination in the Werner states (eq. (1.29)). The specified values of η are the critical values for which the Werner states are guaranteed to exhibit different quantum correlations: entanglement ($\eta > \frac{1}{3}$), steering ($\eta > \frac{1}{2}$), and nonlocality($\eta > 1/K_G(3)$).

upper bound of $\frac{1}{\sqrt{2}} \approx 0.7071$ follows from the well-known CHSH inequality [39]. In 2008, Vértesi [50] calculated a better upper bound to $1/K_G(3)$, 0.7056, by constructing a Bell inequality that is violated by the Werner state with $\eta = 0.7056$ by performing 465 measurements on each side. On 2015, Hua *et al.* [51] calculated the upper bound of 0.7054, and it is the best upper bound known to date to the extent of our knowledge. As for lower bounds, a result by Krivine [52] from 1979 implies the best known lower bound to date of 0.6595. To summarize:

- For $\eta > \frac{1}{3}$ $\rightarrow \rho_W$ is **entangled**.
- For $\eta > \frac{1}{2}$ $\rightarrow \rho_W$ is **steerable**.
- For $\eta > \frac{1}{K_G(3)} \rightarrow \rho_W$ is **nonlocal**, $\frac{1}{K_G(3)} \in [0.6595, 0.7054]$.

Notice that the value $\eta = \frac{1}{2}$ bellow which there exists an LHS model for the Werner states takes into consideration all projective measurements. This means that the scenario in which the Werner states with $\eta = \frac{1}{2}$ are unsteerable allows an *infinite* number of sets of measurements. But what if Alice is constrained to perform a finite number of measurements? Or allowed to perform general measurements? Little is known about the critical value of η in these scenarios. The original results of this dissertation, presented in chapter 4, go in the direction of filling this gap.

CHAPTER 2

More on Quantum Steering

This chapter is devoted to presenting some interesting and useful results in the theory of quantum steering. We formalize the phenomenon of one-way steering, which arises from the asymmetrical nature of steering, and also discuss the role of local operations in steering and entanglement detection. Next, we explore the relationship between steering and joint measurability, showing the intimate connection between measurement incompatibility and steerability. Afterwards, we define quantifiers of steerability for assemblages and quantum states. At last, we finish this chapter by presenting references to other major results in steering theory which are not discussed in this text, such as multipartite steering, hidden steerability, postquantum steering, and applications to information processing and quantum cryptography.

2.1 One-way steering

A characteristic feature of quantum steering is the asymmetry in the roles played by each party. This is highlighted by the approach we use to the treatment of the information that comes from each side: Alice is untrusted, device-independent, and described by conditional probability distributions, while Bob is trusted, device-dependent, and described by a quantum state.

This asymmetry distinguishes steering from entanglement and Bell nonlocality, where the roles of the observers can be interchanged without any effect on the perceived correlation. Steering, however, is directional. The fact that Alice is able to steer Bob does not imply that Bob is able to steer Alice. In fact, for some states, even if it is possible for one party to steer the other (A→B), steering is not possible in the opposite direction (B↛A). This phenomenon, called **one-way steering**, was first proposed theoretically by reference [53] for continuous variable systems and Gaussian measurements and experimentally verified on reference [54].

More recently on reference [40], the authors presented an entangled two-qubit state that can be steered from Alice to Bob (A→B) but not from Bob to Alice (B↛A), by performing projective measurements on each party. Let us present their example of a one-way steerable state.

Consider the following family of two-qubit states:

$$\rho_{AB}(\alpha) = \alpha\,|\Psi^-\rangle\langle\Psi^-| + \frac{(1-\alpha)}{5}\left(2\,|0\rangle\langle 0| \otimes \frac{\mathbb{1}}{2} + 3\,\frac{\mathbb{1}}{2} \otimes |1\rangle\langle 1|\right) \tag{2.1}$$

where $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ is the singlet state and $\alpha \in [0,1]$. The authors have shown that for $0.4983 \lesssim \alpha \leq \frac{1}{2}$ these states are one-way steerable from Alice to Bob for projective measurements.

Let us start by showing that steering is not possible from Bob to Alice (B↛A) if Bob performs local projective measurements on his part of the quantum state $\rho_{AB}(\frac{1}{2})$. In order to do so, we explicitly construct an LHS model for the assemblage $\{\sigma_{b|y}\}$ held by Alice that is generated when Bob locally performs projective measurements $\{M_{b|y}\}$ on state $\rho_{AB}(\frac{1}{2})$. Let $M_{b|y} = \frac{1}{2}(\mathbb{1} + b\,\vec{y}\cdot\vec{\sigma})$ be Bob's measurement elements, $b \in \{\pm 1\}$ and $\vec{y}$ be a unit 3-vector, and $\vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ be the vector of Pauli matrices. An LHS model for an assemblage held by Alice has the form:

$$\sigma_{b|y} = \sum_{\lambda} \pi(\lambda)\, p_B(b|y,\lambda)\, \rho_\lambda, \quad \lambda \in \Lambda. \tag{2.2}$$

The proposed LHS model works as follows. The classical hidden variable is a 4-vector

$$\lambda = (\lambda_0, \vec{\lambda}), \tag{2.3}$$

where $\lambda_0 \in \{\pm 1\}$ and $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is a unit 3-vector. The probability distribution of the values of the LHV is given by

$$\pi(\lambda) = p(\lambda_0)\omega(\vec{\lambda}), \tag{2.4}$$

where the probability of λ_0 assuming the values ± 1 is $p(\lambda_0 = +1) = 4/5$ and $p(\lambda_0 = -1) = 1/5$ while the 3-vector $\vec{\lambda}$ is distributed according to the probability density $\omega(\vec{\lambda}) = \frac{1}{4\pi}(1 + \lambda_3)$. Whenever the hidden variable Λ assumes value λ and Bob chooses to perform the projective measurement $\{M_{b|y}\}$, his response function $p_B(b|y, \lambda)$ is deterministic and he will output the result b according to

$$b = -\lambda_0 \,\mathrm{sign}(\vec{y} \cdot \vec{\lambda}). \tag{2.5}$$

Then, Alice's local hidden state ρ_λ will be a pure qubit state of the form $\rho_\lambda = \frac{1}{2}(\mathbb{1} + \lambda_0 \vec{\lambda} \cdot \vec{\sigma})$.

In reference [40], it is shown that this LHS model recovers the statistics for any arbitrary projective measurements performed by Bob on his share of the state $\rho_{AB}(\frac{1}{2})$, and thus it is proven that this state is unsteerable from Bob to Alice. To decrease α is to increase the share of the separable state in the decomposition of the states $\rho_{AB}(\alpha)$, which cannot increase the steerability of states $\rho_{AB}(\alpha)$. By consequence, all states $\rho_{AB}(\alpha)$ with $\alpha \leq \frac{1}{2}$ are unsteerable from Bob to Alice for projective measurements.

To show that for $\alpha \approx 0.4983$, the state $\rho_{AB}(\alpha)$ is steerable from Alice to Bob (A$\to$B), the authors ran a series of optimization programs for different fixed numbers of projective measurements. These programs output a set of measurements that demonstrates steering for a given value of α. They found a set of 14 projective measurements that demonstrates steering when performed on Alice's share of $\rho_{AB}(\approx 0.4983)$ and then constructed a steering inequality that witnesses this phenomenon [40]. Thus, it was proven that for $\alpha \gtrsim 0.4983$ the state $\rho_{AB}(\alpha)$ is steerable from Alice to Bob. This is the lowest value of α for which $\rho_{AB}(\alpha)$ is steerable from Alice to Bob that was reported.

Since for $\alpha \leq 1/2$ the states $\rho_{AB}(\alpha)$ are unsteerable from Bob to Alice (B$\not\to$A) and for $\alpha \gtrsim 0.4983$ they are steerable from Alice to Bob (A$\to$B), it is proven that all states $\rho_{AB}(\alpha)$ with $\alpha \in [0.4983, \frac{1}{2}]$ are one-way steerable for projective measurements.

But what about for general POVMs? In a following paper [44], the authors presented a method for constructing one-way steerable states for POVMs from one-way steerable states for projective measurements. This result follows from two other important results in the theory of quantum steering, demonstrated in the same article, that will be presented in the following.

The first important result of reference [44] concerns the construction of LHS models for general POVMs acting on quantum states from LHS models for projective measurements acting on quantum states.

Lemma 1. *Let ρ_{AB} be a bipartite entangled state, shared by Alice and Bob, acting on the Hilbert space $\mathcal{H}^d \otimes \mathcal{H}^d$. Suppose ρ_{AB} admits an LHS model from Bob to Alice for projective measurements, and let ρ_A be the reduced state of Alice and $|d+1\rangle\langle d+1|$ be a projector onto a subspace that is orthogonal to the support of ρ_B, the reduced state of Bob. Then,*

$$\rho'_{AB} = \frac{1}{d+1}\left(\rho_{AB} + \rho_A \otimes d|d+1\rangle\langle d+1|\right) \tag{2.6}$$

is a bipartite entangled state acting on $\mathcal{H}^d \otimes \mathcal{H}^{d+1}$ that admits an LHS model from Bob to Alice for POVMs.

This theorem is presented on reference [44] and its proof explores the methods of reference [55]. It can be seen as a particular case of a result in reference [55] for nonlocality, where starting from a state that has an LHV model for projective measurements, the authors present a protocol to construct a state that admits an LHV model for POVMs.

The second important result of [44] concerns the fact that it is not possible to generate steerability with local operations performed by the steered party.

First, let us define some positivity conditions for linear maps which will be recurrently used.

Definition 2.1 (Positive, k-positive and completely positive maps)**.** *Let $\Lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be a linear map and let $A \in \mathcal{L}(\mathcal{H})$ and $A_k \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}_k)$. Then,*
Λ is a positive map if

$$A \geq 0 \implies \Lambda(A) \geq 0. \tag{2.7}$$

Λ is a k-positive map if

$$A_k \geq 0 \implies (\Lambda \otimes \mathbb{1}_k)(A_k) \geq 0, \tag{2.8}$$

that is, its trivial k-extension is a positive map.
Λ is a completely positive (CP) map if it is k-positive for all $k \in \mathbb{N}$.

Furthermore, a linear map is trace preserving when $\mathrm{Tr}(\Lambda(A)) = \mathrm{Tr}(A)$.

Lemma 2. *Let ρ_{AB} be a bipartite state such that any assemblage $\{\sigma_{a|x}\}$, $\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\,\rho_{AB})$ resulting from local measurements performed by Alice on ρ_{AB} is unsteerable from Alice to Bob. For any local operation represented by an arbitrary positive map Λ acting on Bob's side, the assemblage $\tilde{\sigma}_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\,\tilde{\rho}_{AB})$ resulting from local measurements performed by Alice on $\tilde{\rho}_{AB}$, where*

$$\tilde{\rho}_{AB} = \frac{(\mathbb{1} \otimes \Lambda)(\rho_{AB})}{\mathrm{Tr}\left[(\mathbb{1} \otimes \Lambda)(\rho_{AB})\right]}, \tag{2.9}$$

is also unsteerable from Alice to Bob.

We present the proof from reference [44].

Proof. First, notice that an LHS model can be written for continuous local hidden variables, analogously to eq. (1.17), as

$$\sigma_{a|x} = \int \pi(\lambda)\, p_A(a|x,\lambda)\, \rho_\lambda\, d\lambda. \tag{2.10}$$

Since ρ_{AB} admits an LHS model from Alice to Bob, any assemblage $\{\sigma_{a|x}\}$ generated from local measurements $\{M_{a|x}\}$ on Alice's side, given by $\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\,\rho_{AB})$, admits an LHS model. The resulting assemblage $\{\tilde{\sigma}_{a|x}\}$ from Alice performing measurements $\{M_{a|x}\}$ on $\tilde{\rho}_{AB}$, after local operations Λ on Bob's side of ρ_{AB}, given by $\tilde{\sigma}_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\,\tilde{\rho}_{AB})$, is related to $\{\sigma_{a|x}\}$ by

$$\tilde{\sigma}_{a|x} = \frac{1}{\mathrm{Tr}\left[\Lambda(\rho_B)\right]}\,\Lambda(\sigma_{a|x}). \tag{2.11}$$

Substituting in eq. (2.10),

$$\tilde{\sigma}_{a|x} = \int \frac{\pi(\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_B)\right]}\, p_A(a|x,\lambda)\, \Lambda(\rho_\lambda)\, d\lambda \tag{2.12a}$$

$$= \int \frac{\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right]\, \pi(\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_B)\right]}\, p_A(a|x,\lambda)\, \frac{\Lambda(\rho_\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right]}\, d\lambda. \tag{2.12b}$$

Redefining $\tilde{\rho}_\lambda = \dfrac{\Lambda(\rho_\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right]}$, a normalized quantum state, and $\tilde{\pi}(\lambda) = \dfrac{\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right]\, \pi(\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_B)\right]}$, we have

$$\tilde{\sigma}_{a|x} = \int \tilde{\pi}(\lambda)\, p_A(a|x,\lambda)\, \tilde{\rho}_\lambda\, d\lambda. \tag{2.13}$$

We need to show that $\tilde{\pi}(\lambda)$ is a valid probability density.

Since from eq. (1.11) $\sum_a \sigma_{a|x} = \rho_B = \int \pi(\lambda)\rho_\lambda d\lambda$, it follows that

$$\Lambda(\rho_B) = \int \pi(\lambda)\Lambda(\rho_\lambda)d\lambda \tag{2.14a}$$

$$\mathrm{Tr}\left[\Lambda(\rho_B)\right] = \int \pi(\lambda)\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right] d\lambda \tag{2.14b}$$

$$1 = \int \frac{\mathrm{Tr}\left[\Lambda(\rho_\lambda)\right]\ \pi(\lambda)}{\mathrm{Tr}\left[\Lambda(\rho_B)\right]} d\lambda \tag{2.14c}$$

$$1 = \int \tilde{\pi}(\lambda)\, d\lambda. \tag{2.14d}$$

Therefore, eq. (2.13) is indeed an LHS model and $\tilde{\rho}_{AB}$ can only generate unsteerable assemblages from Alice to Bob. □

In practice, this means that steerability cannot be created from local operations on the steered party of an unsteerable quantum state.

It is important to notice that since the local operations Λ performed by Bob are positive maps, instead of completely positive maps, the resulting object $\tilde{\rho}_{AB}$ from eq. (2.9) need not be a valid quantum state, as the operation $\mathbb{1} \otimes \Lambda$ that acts on ρ_{AB} need not be a positive map. Nevertheless, we can still talk about assemblages that result from local measurement on this object $\tilde{\rho}_{AB}$. The reason is that, locally, the reduced state for both parts continues to be a valid quantum state even though, globally, the object used to describe both systems may no longer be a quantum state. Since only local measurements are performed by Alice after the action of Bob's local operations, the resulting assemblage is still valid. Furthermore, in the case where the state ρ_{AB} is unsteerable from Alice to Bob, all possible assemblages resulting from $\tilde{\rho}_{AB}$ are also unsteerable from Alice to Bob.

An interesting corollary of lemma 2 is the following:

Corollary 1. *If the assemblage $\{\tilde{\sigma}_{a|x}\}$ is steerable from Alice to Bob, then $\{\sigma_{a|x}\}$, and consequently ρ_{AB}, is also steerable from Alice to Bob.*

It is possible to restrict the class of local operations from lemma 2 to achieve a complementary result.

Theorem 1. *Let ρ_{AB} be a bipartite state shared by Alice and Bob. If Bob performs local operations on his subsystem described by a completely positive, invertible map Λ, then the resulting state $\tilde{\rho}_{AB}$ (eq. (2.9)) will be unsteerable from Alice to Bob if, and only if, ρ_{AB} is unsteerable from Alice to Bob.*

Proof. If ρ_{AB} is unsteerable from Alice to Bob, the proof of the unsteerability of $\tilde{\rho}_{AB}$ from Alice to Bob follows from the proof of lemma 2. Now, since Λ is a completely positive map, $\tilde{\rho}_{AB}$ is a valid quantum state. Since Λ is invertible and the inverse of a completely positive map is also completely positive, given a state $\tilde{\rho}_{AB}$ we can reconstruct a valid quantum state ρ_{AB}. From the same reasoning in the proof of lemma 2, if $\tilde{\rho}_{AB}$ is unsteerable from Alice to Bob, ρ_{AB} is also unsteerable from Alice to Bob. $\square$

A particular kind of local operations that fits lemma 2 is a local **filter**, a map Λ such that $\Lambda(\rho) = F\rho F^\dagger$ and $F^\dagger F \leq \mathbb{1}$.

As an example of the use of filters in steering, let us consider the following qubit-qutrit state, $E(\alpha) \in \mathcal{D}(\mathcal{H}^2 \otimes \mathcal{H}^3)$, called the *erasure state*:

$$E(\alpha) = \alpha|\Psi^-\rangle\langle\Psi^-| + (1-\alpha)\frac{\mathbb{1}_2}{2} \otimes |2\rangle\langle 2|, \tag{2.15}$$

where $|\Psi^-\rangle\langle\Psi^-|$ here is the two-qubit singlet state embedded in the qubit-qutrit Hilbert space. This state can be seen as the result of sending half of a two-qubit maximally entangled state through the erasure channel [56]. A priori, we do not know whether or not this state is entangled, steerable, or Bell nonlocal.

However, if we apply the filter $F_B = |0\rangle\langle 0| + |1\rangle\langle 1|$, $F_B \in \mathcal{L}(\mathcal{H}^3)$, $F_B^\dagger = F_B$ to Bob's subsystem[1], we will have the resulting state

$$\tilde{E}(\alpha) = \frac{(\mathbb{1} \otimes F_B)E(\alpha)(\mathbb{1} \otimes F_B)}{\mathrm{Tr}\left[(\mathbb{1} \otimes F_B)E(\alpha)(\mathbb{1} \otimes F_B)\right]}. \tag{2.16}$$

With some straightforward calculation, it can be checked that

$$(\mathbb{1} \otimes F_B)E(\alpha)(\mathbb{1} \otimes F_B) = \alpha|\Psi^-\rangle\langle\Psi^-|, \tag{2.17}$$

and $\mathrm{Tr}(\alpha|\Psi^-\rangle\langle\Psi^-|) = \alpha$. Hence, for $\alpha > 0$, the erasure state after filter F_B in Bob is

$$\tilde{E}(\alpha) = |\Psi^-\rangle\langle\Psi^-|, \tag{2.18}$$

the singlet state.

We know that the singlet state is maximally entangled, steerable, and Bell nonlocal. Thus, we can explore lemma 2 to learn about the correlations in the erasure state $E(\alpha)$.

[1]Notice that this filter corresponds to a qubit identity embedded in a qutrit space.

First, if the resulting state after the filter in Bob is steerable from Alice to Bob, necessarily the original state was also steerable from Alice to Bob, for any value of α. For the erasure state to be originally steerable, it is necessary that it was also entangled, for any value of α. But the fact that after the filter the state $\tilde{E}(\alpha)$ is nonlocal does not mean that $E(\alpha)$ was nonlocal, as lemma 2 does not hold for nonlocality. The nonlocality of $E(\alpha)$ depends on the value of α [55].

Finally, we can use lemma 1 and lemma 2 to construct one-way steerable states for POVMs.

Theorem 2. *Let ρ_{AB} be a bipartite entangled state acting on $\mathcal{H}^d \otimes \mathcal{H}^d$ steerable from Alice to Bob but unsteerable from Bob to Alice for projective measurements. Then, the state ρ'_{AB} from eq. (2.6) is also steerable from Alice to Bob but unsteerable from Bob to Alice for POVMs.*

Proof. Since ρ_{AB} has an LHS model from Bob to Alice for projective measurements, it follows from lemma 1 that ρ'_{AB} has an LHS model from Bob to Alice for POVMS. Moreover, there exists a local operation, the filter $F_B = |0\rangle\langle 0| + |1\rangle\langle 1|$, $F_B \in \mathcal{L}(\mathcal{H}^{d+1})$, that allows Bob to map ρ'_{AB} in ρ_{AB}, which by assumption is steerable from Alice to Bob. Hence, it follows from corollary 1 that ρ'_{AB} is steerable from Alice to Bob as well. □

According to theorem 2 we can construct an example of a one-way steerable state for POVMs by applying eq. (2.6) to the state $\rho_{proj} = \rho_{AB}(1/2)$ of eq. (2.1), which is one-way steerable from Alice to Bob for projective measurements. The result is the state

$$\rho_{\text{povm}} = \frac{1}{3}\left(\rho_{\text{proj}} + 2\rho_{\text{proj,A}} \otimes |2\rangle\langle 2|\right), \tag{2.19}$$

where $\rho_{\text{proj,A}} = \text{Tr}_B(\rho_{\text{proj}})$, which is one-way steerable for POVMs from Alice to Bob.

In a recent paper [57], the authors presented a method for constructing simpler one-way steerable states and give an example of a two-qubit state that is unsteerable from Alice to Bob for POVMs and steerable from Bob to Alice with only two measurements, instead of the 14 measurements of reference [40].

Recent experimental demonstrations of one-way steering have been reported in [58] and [59].

2.2 Mapping to joint measurability

Besides nonlocality, another notable aspect of quantum theory is the fact that some quantum measurements cannot be jointly performed on the same physical system. This phenomenon is questioned by EPR [2], when they claim that all elements of the physical reality, according to their definition, such as properties of physical systems like position and momentum, should have a counterpart in any complete physical theory. Being their reality simultaneous, it should be possible to jointly measure these properties as well. Nowadays, we know measurement incompatibility to be an actual aspect of nature and not a consequence of some possible incompleteness of the theory of quantum mechanics, as was thought by EPR.

This notion is usually captured by the non-commutativity of quantum observables (such as position and momentum or different spin directions), which is appropriate for projective measurements. However, there exist quantum measurements that are not represented by observables, the non-projective POVMs, that also demonstrate incompatibility, in the sense that they too cannot be jointly measured. To capture this more general notion of incompatibility we will use the concept of **joint measurability** defined as follows:

Definition 2.2 (Joint measurability)**.** *A set of N measurements with O outcomes each $\{M_{a|x}\}_{a,x}$, with $a \in \{1,\dots,O\}$ and $x \in \{1,\dots,N\}$ is jointly measurable if there exists a set of operators $\{M_{\vec{a}_k}\}_{\vec{a}_k}$, called mother-POVM, with $k \in \{1,\dots,O^N\}$ outcomes labeled by $\vec{a}_k = (a_{x=1},...,a_{x=O})$ such that*

(i) $M_{\vec{a}_k} \geq 0 \quad \forall\, \vec{a}_k$

(ii) $\sum_{\vec{a}_k} M_{\vec{a}_k} = \mathbb{1}$

(iii) $\sum_{\vec{a}_k|a_x=a} M_{\vec{a}_k} = M_{a|x}, \quad \forall\, x, a.$

Conditions *(i)* and *(ii)* guarantee that $\{M_{\vec{a}_k}\}$ is indeed a valid POVM while condition *(iii)* guarantees that $\{M_{\vec{a}_k}\}$ is "mother" of $\{M_{a|x}\}$, meaning that the measurement elements $M_{a|x}$ can be obtained via coarse-graining of $\{M_{\vec{a}_k}\}$.

Hence, to claim that a set of measurements is jointly measurable is to say that there exists a mother-POVM for the set, *i.e.*, a valid POVM that can jointly determine all possible outcomes of performing each measurement of the original set.

It turns out that incompatibility can be used as a resource, much in the same way as entanglement, steering, and nonlocality [60, 61].

Some effort has been devoted to the study of the relationship between measurement incompatibility and nonlocality [62–65]. It is known that incompatible measurements and entanglement are two necessary ingredients to demonstrate nonlocality. This means that compatible measurements on any quantum state can never lead to nonlocal boxes, neither can incompatible measurements on separable states[2]. Nevertheless, they are not sufficient ingredients: not all incompatible measurements on arbitrary entangled states can demonstrate nonlocality. These results have been proven for projective measurements as well as for general POVMs [66].

On the other hand, the relationship between measurement incompatibility and steering is much tighter. Indeed, as first demonstrated in references [66] and [67], they can be seen as equivalent resources. For all sets of incompatible measurements, there exists an entangled state from which steering can be demonstrated. All steerable assemblages are generated by incompatible measurements on entangled states.

To demonstrate these results we will follow the work of reference [66].

Theorem 3. *The assemblage* $\{\sigma_{a|x}\}$*,* $\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, \rho_{AB})$*, is unsteerable for all arbitrary entangled bipartite state* ρ_{AB} *if, and only if, the set of POVMs* $\{M_{a|x}\}$ *is jointly measurable.*

Proof. Joint measurability implies unsteerability: consider a set of jointly measurable POVMs $\{M_{a|x}\}$ with mother-POVM $\{M_{\vec{a}_k}\}$. The assemblage resulting from these measurements on an arbitrary bipartite state ρ_{AB} admits an LHS model in the form of eq. (1.17) with the local hidden variable assuming values $\lambda = \vec{a}_k$ with probability distribution $\pi(\vec{a}_k) = \mathrm{Tr}(M_{\vec{a}_k}\, \rho_A)$, where ρ_A is the reduced state of Alice. According to λ, Bob's conditioned states will be $\rho_{\vec{a}_k} = \mathrm{Tr}_A(M_{\vec{a}_k} \otimes \mathbb{1}\, \rho_{AB})$ and Alice will announce

[2]Trivially, compatible measurements on separable states will never lead to nonlocality.

outcome a according to the deterministic response function $p_A(a|x, \vec{a}_k) = \delta_{a,a_x}$. Hence, the resulting assemblage is unsteerable.

Unsteerability implies joint measurability: Consider the unsteerable assemblage $\{\sigma_{a|x}\}$, resulting from local measurements $\{M_{a|x}\}$ on the bipartite pure[3] maximally entangled state $\rho_{AB} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$, given by

$$\begin{aligned}
\sigma_{a|x} &= \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}|\psi\rangle\langle\psi|) && (2.20\text{a})\\
&= \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\,\frac{1}{d}\sum_{i,j}|ii\rangle\langle jj|) && (2.20\text{b})\\
&= \mathrm{Tr}_A(\mathbb{1} \otimes M_{a|x}^T\,\frac{1}{d}\sum_{i,j}|ii\rangle\langle jj|) && (2.20\text{c})\\
&= \frac{1}{d}\sum_{i,j}\mathrm{Tr}_A(|i\rangle\langle j| \otimes M_{a|x}^T|i\rangle\langle j|) && (2.20\text{d})\\
&= \frac{1}{d}M_{a|x}^T, && (2.20\text{e})
\end{aligned}$$

with $(\cdot)^T$ denoting the transpose. The passage from eq. (2.20b) to eq. (2.20c) follows from the fact that (droping the label $a|x$ from the POVM elements) $M|i\rangle = \sum_i M_{ij}|j\rangle$ and hence we can write

$$M\otimes\mathbb{1}|\psi\rangle = \sum_i M\otimes\mathbb{1}|ii\rangle = \sum_{i,j} M_{ij}|j\rangle\otimes|i\rangle = \sum_{i,j}(M_{ji})^T|ji\rangle = \sum_j \mathbb{1}\otimes M^T|jj\rangle = \mathbb{1}\otimes M^T|\psi\rangle. \tag{2.21}$$

Now let us define

$$\sigma_{\vec{a}_k} = \sum_\lambda \pi(\lambda)\,\tilde{p}_A(\vec{a}_k|\lambda)\,\rho_\lambda, \tag{2.22}$$

such that $\sum_{\vec{a}_k|a_x=a}\tilde{p}_A(\vec{a}_k|\lambda) = p_A(a|x,\lambda)$. Notice that $\sum_{\vec{a}_k|a_x=a}\sigma_{\vec{a}_k} = \sigma_{a|x}$. Defining

$$M_{\vec{a}_k} = d\,\sigma_{\vec{a}_k}^T, \tag{2.23}$$

it can be checked that $\{M_{\vec{a}_k}\}$ is a valid mother-POVM to $\{M_{a|x}\}$. Hence, the measurement set is jointly measurable. The extension for assemblages resulting from measurements on partly entangled pure states comes from theorem 1: if the resulting assemblage from measurements on the maximally entangled state is unsteerable, applying a filter on Bob's

[3]We can assume the quantum state to be pure since we are not restricting the dimension of the Hilbert space on which it is acting. This follows from Naimark's theorem [68].

side that will take the maximally entangled state to partly entangled states will preserve the unsteerability of the state. □

Another way of looking at this relationship is to understand the connection between LHS models and mother-POVMs. This will be presented following the work of reference [69]. Let us begin by rewritting the coarse-graining condition in definition 2.2(iii) as

$$M_{a|x} = \sum_{\lambda} D(a|x,\lambda) M_{\vec{a}_\lambda}, \tag{2.24}$$

where $D(a|x,\lambda)$ are deterministic probability distributions and $\{M_{\vec{a}_\lambda}\}$ is a mother-POVM for $\{M_{a|x}\}$. This definition is equivalent to the previous one and it allows us to physically interpret the measurement element $M_{a|x}$ as performing the measurement $\{M_{\vec{a}_\lambda}\}$ and post-processing the obtained probabilities. At this point, it is convenient to recall that LHS description of an assemblage can be written in the form of eq. (1.20):

$$\sigma_{a|x} = \sum_{\lambda} D(a|x,\lambda)\, \sigma_\lambda. \tag{2.25}$$

We proceed to show how to obtain LHS models from mother-POVMs and vice versa.

Given a bipartite quantum state ρ_{AB}, let Π_B be a projection on the subspace of the image of the reduced state ρ_B, *i.e.*, $\Pi_B : \mathcal{H}_B \to \mathcal{K}_B$, where $\mathcal{K}_B \coloneqq \text{range}(\rho_B) \subset \mathcal{H}_B$ is the subspace of $\mathcal{H}_B$ that corresponds to the image of ρ_B, Π_B satisfies $\Pi_B \Pi_B^\dagger = \mathbb{1}_{\mathcal{K}_B}$, and $\Pi_B^\dagger \Pi_B \in \mathcal{L}(\mathcal{H}_B)$ is a Hermitian projector. Defining the full-rank operator $\tilde{\rho}_B = \Pi_B \rho_B \Pi_B^\dagger \in \mathcal{K}_B$, we preserve the positivity of the operator $\tilde{\rho}_B$ and guarantee that the inverse $\tilde{\rho}_B^{-1}$ exists[4].

Let $\{B_{a|x}\} \in \mathcal{K}_B$ be the so-called steering-equivalent (SE) POVM defined by

$$B_{a|x} = \tilde{\rho}_B^{-\frac{1}{2}}\, \sigma_{a|x}\, \tilde{\rho}_B^{-\frac{1}{2}}. \tag{2.26}$$

Notice that $B_{a|x} \geq 0$ and $\sum_a B_{a|x} = \mathbb{1}_{\mathcal{K}_B}$, so it is indeed a POVM. Now we are ready to state the following theorem:

Theorem 4. *The assemblage $\{\sigma_{a|x}\}$ is unsteerable if, and only if, the corresponding steering-equivalent POVM $\{B_{a|x}\}$ is jointly measurable.*

[4]The reason why the definition of $\tilde{\rho}_B$ is necessary is that we will be working with the inverse of this operator. For the reduced state ρ_B the inverse does not always exists since ρ_B could be, for example, a pure state which is not full-rank and thus is not invertible. By constructing the full-rank operator $\tilde{\rho}_B$ we guarantee it is invertible and the map $\tilde{\rho}_B^{-\frac{1}{2}}$ always exists.

Proof. It is sufficient to see that from the LHS model of the unsteerable assemblage $\{\sigma_{a|x}\}$ in eq. (2.25) we can construct a mother-POVM $\{B_{\vec{a}_\lambda}\}$ for $\{B_{a|x}\}$ according to

$$B_{\vec{a}_\lambda} = \tilde{\rho}_B^{-\frac{1}{2}} \, \sigma_\lambda \, \tilde{\rho}_B^{-\frac{1}{2}}, \tag{2.27}$$

and that from a mother-POVM $\{B_{\vec{a}_\lambda}\}$ to $\{B_{a|x}\}$ we can construct an LHS model for $\{\sigma_{a|x}\}$ according to

$$\sigma_\lambda = \tilde{\rho}_B^{\frac{1}{2}} \, B_{\vec{a}_\lambda} \, \tilde{\rho}_B^{\frac{1}{2}}. \tag{2.28}$$

□

One great advantage of the one-to-one mapping between steering and joint measurability is that now it is possible to apply known results of joint measurability theory to steering theory, and the other way around. Moreover, new results from both areas will be directly applicable to each other.

Some important observations that follow from these results are:

(i) Every problem of jointly measurability of a set of measurements can be mapped into a problem of steerability of an assemblage.

(ii) An LHS model for an unsteerable assemblage can be mapped into a parent-POVM of a set of jointly measurable measurements, and vice versa.

(iii) A set of POVMs is not jointly measurable if and only if it can be used to demonstrate steering.

2.3 Quantifying steering

If the first question one could ask when hoping to use steering as a resource in quantum information tasks is whether or not a given assemblage or quantum state is steerable, the next natural question should be: *how* steerable is it?

The problem of quantifying steerability is not an easy one. There is no single quantifier and different definitions will return different quantifications for the same objects [70]. Moreover, there is no closed analytical formula to quantify steerability even in the simplest cases (low dimensions and few measurements) [10]. Most methods depend heavily on numerical calculations and can demand large amounts of computational power.

In this section we will present three steering quantifiers: steering weight [26], generalized robustness of steering [70], and white noise robustness of steering. For now we will only focus on their definitions. On chapter 3 we will explore some techniques for actually calculating the value of these three quantifiers for assemblages and for quantum states.

2.3.1 Steering weight

The steering weight was defined by Paul Skrzypczyk, Miguel Navascués, and Daniel Cavalcanti on reference [26]. As they point out in their paper, this steering quantifier has an operational motivation behind it. Let us say Alice is going to prepare a determined steerable assemblage for Bob. In doing so, she is going to minimize the use of the steerable resource that is available to her. As often as she can, Alice will prepare an unsteerable assemblage for Bob and only at times she will prepare an actual steerable assemblage, in such a way that, on average, she prepares the assemblage she desires. This means that we can decompose the assemblage prepared for Bob as

$$\sigma_{a|x} = \mu\, \pi^{\text{uns}}_{a|x} + (1-\mu)\, \pi^{\text{ste}}_{a|x}, \quad \mu \in [0,1], \tag{2.29}$$

where $\{\pi^{\text{uns}}_{a|x}\} \in \mathcal{U}$ is an unsteerable assemblage prepared with probability μ and $\{\pi^{\text{ste}}_{a|x}\} \notin \mathcal{U}$ is a steerable assemblage prepared with probability $1-\mu$. The **steerable weight** sw of the assemblage $\{\sigma_{a|x}\}$ is given by

$$sw = 1 - \mu^*, \tag{2.30}$$

where μ^* is the maximal value of μ so that eq. (2.29) holds.

To calculate sw we must optimize over all possible convex decompositions of the assemblage $\{\sigma_{a|x}\}$ and find the assemblages $\{\pi^{\text{uns}}_{a|x}\}$ and $\{\pi^{\text{ste}}_{a|x}\}$ that maximize μ. The steering weight is the minimum value that $(1-\mu)$ can assume. It is the minimum amount of the steering resource that is necessary to reproduce $\{\sigma_{a|x}\}$.

Assemblages with maximal steering weight will be called maximally steerable for SW. However, this does not mean that they will also be maximally steerable for other steering quantifiers [70]. Yet, any unsteerable assemblage will have $sw = 0$, since no steerable resource is necessary to prepare them, and this is expected to hold for any steering quantifier.

One property of the steering weight is that all pure entangled states are maximally steerable for SW (even the ones that have very little entanglement).

The steering weight of the two-qubit ($d = 2$) Werner state (eq. (1.29)) is maximal for $\eta = 1$ (where the state is also maximally entangled) and decreases monotonically with η. Werner states with dimension larger than 2 and $\eta = 1$ are also maximally steerable for SW. This is interesting because there is no evidence that these states are nonlocal, in particular, unlike the case of $d = 2$, no Bell inequality that is violated by these states is known [26]. This makes them good candidates for maximally entangled and maximally steerable, yet local, states.

The steering weight of an assemblage can be calculated by semidefinite programming [26] in an efficient way, as will be further discussed in section 3.2. However, there do not exist general analytical solutions.

The steering weight is analogous to the entanglement quantifier *best separable approximation* [71], an optimal decomposition of a quantum state in separable and entangled parts, and analogous to the nonlocality quantifier *EPR2* [72], an optimal decomposition of probability distributions that works in the same way.

2.3.2 Generalized robustness of steering

The robustness of steering of an assemblage is a steering quantifier defined in reference [70]. For now, we will present a slightly different definition, that has the same idea behind it but is a more intuitive way to think about this quantifier, and we will call it generalized robustness of steering.

The idea of *robustness*, in general, is to check how much disturbance a system can endure and still be able to demonstrate a certain property [73]. For the case of steering, the robustness of the steerability of an assemblage measures how much the assemblage of interest can be mixed with some other assemblage before it becomes unsteerable. Of course, if the assemblage is unsteerable to begin with, it is not necessary to mix it with anything to make it unsteerable. It is said that these assemblages have no robustness. The more robust an assemblage is, the more it should be possible to mix it before it becomes unsteerable. However, this depends heavily on the object with which the assemblage is being mixed.

Let $\{\gamma_{a|x}\}$ be an assemblage given by the convex combination of the assemblage of interest $\{\sigma_{a|x}\}$ and some other general assemblage $\{\pi_{a|x}\}$,

$$\gamma_{a|x} = p\,\sigma_{a|x} + (1-p)\pi_{a|x}, \quad p \in [0,1]. \tag{2.31}$$

The **generalized robustness of steering** gr of $\{\sigma_{a|x}\}$ is

$$gr = 1 - p^*, \tag{2.32}$$

where p^* is the maximum value that p can assume, optimizing over all assemblages $\{\pi_{a|x}\}$, for which $\{\gamma_{a|x}\} \in \mathcal{U}$ is unsteerable. It is important to notice that we have not made any assumptions on $\{\pi_{a|x}\}$ other than that it is a valid assemblage. It does not have to be unsteerable. The set of steerable assemblages is not convex and this means that by making a convex combination of two steerable assemblages, it could result on an unsteerable assemblage. Indeed, there will be cases for some $\{\sigma_{a|x}\}$ where the assemblage $\{\pi_{a|x}\}$ that maximizes p is steerable.

Unlike the steering weight, the generalized robustness is not a decomposition of an assemblage into unsteerable and steerable parts. Instead, it is a mixture of the assemblage in such a way that it becomes unsteerable. Still unlike steering weight, minimally entangled pure states, which would have maximal steering weight, have small generalized robustness. But in the same way of the steering weight, the generalized robustness of an assemblage is solved by semidefinite programming and there are no simple analytical solutions for it [70].

The generalized robustness of steering is inspired in the generalized robustness of entanglement [74], an entanglement quantifier defined much in the same way.

Other quantifiers for steerability can be defined by imposing restrictions on the nature of $\{\pi_{a|x}\}$. We could ask for $\{\pi_{a|x}\}$ to be unsteerable, hence the optimization would go over the set $\mathcal{U}$. This would be the definition of the *LHS-robustness of steering* [10]. There could also be imposed that $\{\pi_{a|x}\}$ corresponds to a specific assemblage, such as $\{\pi_{a|x}\} = \{\mathbb{1}_d/(dO_A)\ \forall a, x\}$, the unsteerable assemblage where all elements are equally proportional to the identity. This would be the *random robustness of steering*[5] [10]. Each

[5]These steering quantifiers also have peers in entanglement quantification, such as the random robustness of entanglement (analogous to the random robustness of steering) and the absolute robustness of entanglement (analogous to the LHS-robustness of steering) [72].

definition can have different physical interpretations and some may have no interpretation at all. The choice of which quantifier to use depends on the situation and each one will reveal a different aspect of steerability that can be more or less relevant to the problem at hand. On the next section we will define the *white noise robustness of steering*, a steering quantifier with direct physical interpretation and high experimental interest.

2.3.3 White noise robustness of steering

The white noise robustness of steering is a specific notion of robustness of an assemblage that measures how much "white noise" the assemblage can tolerate before it becomes unsteerable.

Experimentally, white noise is the usual model for unbiased disturbances on physical systems or measurement apparatuses[6]. If there are any disturbances to an experiment whose characterization is not known or there is no reason to assume they are biased in a way or another, they can be treated as white noise.

Mathematically, the effect of white noise will be given by mixing the object of interest with a maximally mixed state, a maximally mixed assemblage, or a uniform probability distribution. Hence, for steering, white noise will be represented by the assemblage whose elements are some multiple of the identity operator. Nevertheless, white noise robustness of steering is not equivalent to the random robustness of steering [61], as we will see in a bit.

It is defined as follows: let $\{\gamma_{a|x}\}$ be the resulting assemblage from applying white noise to the assemblage of interest $\{\sigma_{a|x}\}$ of dimension d, given by

$$\gamma_{a|x} = \eta\,\sigma_{a|x} + (1-\eta)\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}_d}{d} \quad \eta \in [0,1], \tag{2.33}$$

where $\mathbb{1}_d$ is the identity operator of dimension d and η is the parameter of the convex combination. The **white noise robustness** wnr of the assemblage $\{\sigma_{a|x}\}$ is given by

$$wnr = 1 - \eta^* \tag{2.34}$$

where η^* is the maximal value for which $\{\gamma_{a|x}\} \in \mathcal{U}$ is unsteerable.

[6]The modeling of white noise is only well defined for finite-dimensional systems.

Compared to the generalized robustness, the difference is that the mixing assemblage $\{\pi_{a|x}\}$ in the definition of the white noise robustness has elements equal to $\mathrm{Tr}(\sigma_{a|x})\mathbb{1}_d/d$. The advantage of this definition, when compared to the random robustness of steering, is that the resulting assemblage, called the **noisy assemblage**, can be seen as the result of the effect of a *quantum channel* acting on the assemblage of interest. So let us begin by formalizing the concept of quantum channels.

Definition 2.3 (Quantum channel)**.** *A quantum channel $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a linear map of operators in the Hilbert space onto operators in the Hilbert space that respects two conditions:*

(i) Φ *is trace-preserving.*

(ii) Φ *is completely positive.*

An exclusive property of CP linear maps is that they can be written in terms of Kraus operators $\{K_i\}$ [75],

$$\Phi(A) = \sum_i K_i A K_i^\dagger, \tag{2.35}$$

and a property of completely positive trace-preserving (CPTP) linear maps is that their Kraus operators satisfy

$$\sum_i K_i^\dagger K_i = \mathbb{1}. \tag{2.36}$$

The demonstration of these properties can be found in reference [76].

The channel in which we are interested at this point is the **quantum depolarizing channel** Λ^η, a linear, trace-preserving, completely positive map that acts on operators in a finite-dimensional Hilbert space:

$$A \mapsto \Lambda^\eta(A) = \eta A + (1-\eta)\mathrm{Tr}(A)\frac{\mathbb{1}_d}{d}, \quad A \in \mathcal{L}(\mathcal{H}^d). \tag{2.37}$$

It is clear why the map should be trace-preserving. When acting on quantum states, the resulting operator must be a quantum state as well. The depolarizing channel is also a unital channel, meaning that it maps the identity operator onto the identity operator itself. This is interesting from the point of view of quantum operations since it is a map that preserves the maximally mixed element of the set.

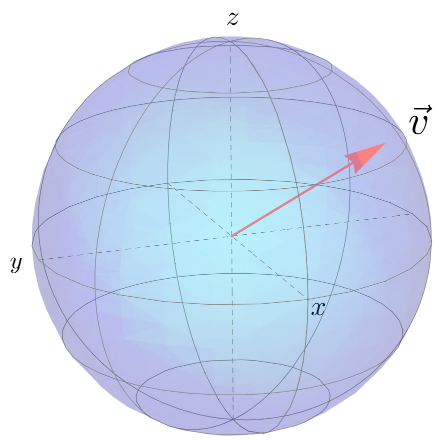

Figure 2.1: The Bloch sphere representation of a qubit state.

When acting on a finite dimensional quantum state ρ, the effect of the depolarizing channel is

$$\rho \mapsto \Lambda^{\eta}(\rho) = \eta\rho + (1-\eta)\frac{\mathbb{1}_d}{d}, \quad \rho \in \mathcal{D}(\mathcal{H}^d) \tag{2.38}$$

since $\mathrm{Tr}(\rho) = 1$.

There is an interesting geometric representation of the action of the depolarizing channel on a two-dimensional quantum state. In order to work with it, it is necessary to get familiar with the representation of qubits on the Bloch sphere.

Any qubit state can be written as

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{v}\cdot\vec{\sigma}\right), \tag{2.39}$$

where $\vec{v}$ is a 3-vector called Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

Since the Pauli matrices are traceless, the factor of $\frac{1}{2}$ multiplying the identity guarantees that ρ has trace 1. To guarantee the positivity of ρ, it is necessary and sufficient that the condition $\|\vec{v}\| \leq 1$ is satisfied. Since there are only three degrees of freedom on a qubit state, each Bloch vector represents a qubit state and all qubit states can be represented by a Bloch vector. Pure qubit states are the ones that have $\|\vec{v}\| = 1$ and mixed qubit states have $\|\vec{v}\| < 1$.

Therefore, the set of all qubit states can be thought of as a ball delimited by a unit 2-sphere, the Bloch sphere, whose respective basis vectors in $\mathbb{R}^3$ correspond to the projectors onto the eigenspaces of the positive eigenvalues of each of the three Pauli matrices. A Bloch sphere and a Bloch vector are represented in fig. 2.1.

The notion of orthogonality in the Bloch sphere differs from that of Euclidian space. Orthogonal qubit states are represented by antipodal vectors, instead of orthogonal vectors. For example, the σ_z eigenstate $|0\rangle$ corresponds to the Bloch vector $\vec{v} = (0, 0, 1)$. The state orthogonal to it, the eigenstate $|1\rangle$ corresponds to the Bloch vector $\vec{v} = (0, 0, -1)$, antipodal to the one of its orthogonal counterpart.

When acting the depolarizing channel on a qubit state, the result is the following:

$$\Lambda^\eta(\rho) = \eta \frac{1}{2}\left(\mathbb{1} + \vec{v} \cdot \vec{\sigma}\right) + (1 - \eta)\frac{\mathbb{1}}{2} \tag{2.40a}$$

$$= \frac{1}{2}\left(\mathbb{1} + \eta\, \vec{v} \cdot \vec{\sigma}\right). \tag{2.40b}$$

Since $\eta \in [0, 1]$, the effect of the depolarizing channel on a qubit is to decrease the norm of the Bloch vector[7], taking $\vec{v} \to \eta\, \vec{v}$. For this reason, the parameter η is known as the **shrinking factor**. The effect of Λ^η when applied to all qubit states can be seen as "the shrinking of the Bloch sphere", as represented on fig. 2.2.

For bipartite states, it is possible to analyze the effect of the depolarizing channel acting locally on each part. Let us say, for example, that Bob prepares many copies of a bipartite system in quantum state ρ_{AB} and sends one half of each copy to Alice through a noisy channel characterized by the map Λ^η. Their global state will be

$$\rho_{AB} \mapsto (\Lambda^\eta \otimes \mathbb{1})(\rho_{AB}). \tag{2.41}$$

Likewise, if Bob's systems are the ones that suffered the action of the depolarizing channel,

$$\rho_{AB} \mapsto (\mathbb{1} \otimes \Lambda^\eta)(\rho_{AB}). \tag{2.42}$$

Now it is clear the importance of a completely positive map: the operation should return a valid quantum state (positive semidefinite operator) even when acting on a

[7] Other kinds of noisy channels have different experimental relevance and different effects on the Bloch sphere, when acting on qubit states. Some examples are the bit-flip channel, the phase-flip channel, the amplitude damping channel, and the phase dumping channel. For an introduction on quantum noise and quantum operations we recommend chapter 8 of reference [4].

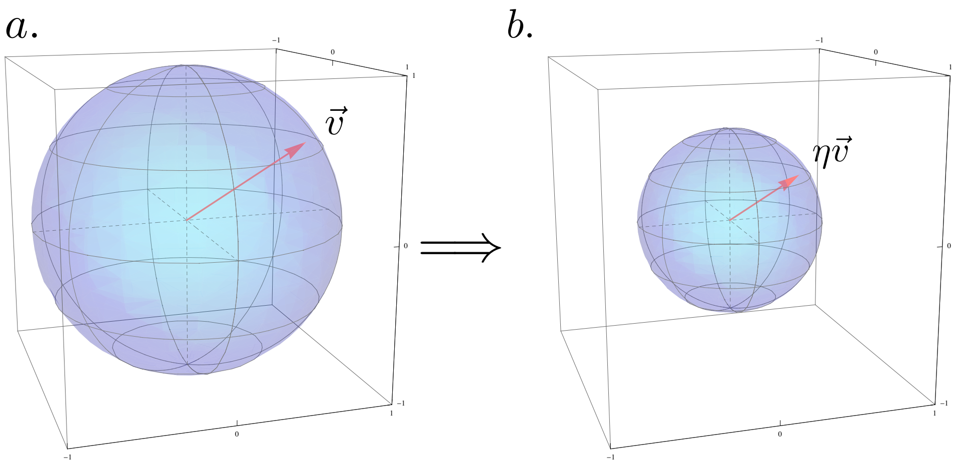

Figure 2.2: The shrinking of the Bloch sphere by action of the depolarizing channel $\Lambda^{\eta}(\rho)$ on qubit states.

single part of the quantum state. The resulting states are the following:

$$(\Lambda^{\eta} \otimes \mathbb{1})(\rho_{AB}) = \eta\rho_{AB} + (1-\eta)\frac{\mathbb{1}_A}{d_A} \otimes \rho_B \tag{2.43a}$$

$$(\mathbb{1} \otimes \Lambda^{\eta})(\rho_{AB}) = \eta\rho_{AB} + (1-\eta)\rho_A \otimes \frac{\mathbb{1}_B}{d_B}, \tag{2.43b}$$

where $\rho_{B(A)} = \mathrm{Tr}_{A(B)}(\rho_{AB})$ is the reduced state for Bob(Alice) and $d_{A(B)}$ is the dimension of Alice(Bob)'s subspace. Notice that after applying the depolarizing map locally on one part of the bipartite state ρ_{AB}, the reduced state of the other party is preserved:

$$\mathrm{Tr}_A\left[(\Lambda^{\eta} \otimes \mathbb{1})(\rho_{AB})\right] = \eta\rho_B + (1-\eta)\mathrm{Tr}\left(\frac{\mathbb{1}_A}{d_A}\right)\rho_B = \rho_B \tag{2.44a}$$

$$\mathrm{Tr}_B\left[(\mathbb{1} \otimes \Lambda^{\eta})(\rho_{AB})\right] = \eta\rho_A + (1-\eta)\rho_A\mathrm{Tr}\left(\frac{\mathbb{1}_B}{d_B}\right) = \rho_A. \tag{2.44b}$$

Let us first analyze an assemblage resulting from local measurements on a bipartite state in the case where Alice's subsystems have been through a depolarizing channel:

$$\sigma^{\eta_A}_{a|x} = \mathrm{Tr}_A\Big(M_{a|x} \otimes \mathbb{1}_B\ (\Lambda^{\eta} \otimes \mathbb{1})(\rho_{AB})\Big). \tag{2.45}$$

Being the depolarizing channel Λ^{η} a CPTP linear map, it can be decomposed into Kraus operators $\{X_k\}$ in the form of eq. (2.35). Accordingly, its trivial extension $\Lambda^{\eta} \otimes \mathbb{1}$ can be written as

$$(\Lambda^{\eta} \otimes \mathbb{1})(A) = \sum_k (X_k \otimes \mathbb{1}_B)A(X_k^{\dagger} \otimes \mathbb{1}_B). \tag{2.46}$$

Substituting in eq. (2.45),

$$\sigma^{\eta_A}_{a|x} = \mathrm{Tr}_A \left(M_{a|x} \otimes \mathbb{1}_B \, \sum_k (X_k \otimes \mathbb{1}_B) \, \rho_{AB} \, (X_k^\dagger \otimes \mathbb{1}_B) \right). \tag{2.47}$$

By writing the bipartite state in the form of $\rho_{AB} = \sum_{i,j} r_{ij} \, A_i \otimes B_j$, where $\{A_i, B_j\}$ form a basis for operators in the Hilbert space,

$$\begin{align}
\sigma^{\eta_A}_{a|x} &= \mathrm{Tr}_A \left(M_{a|x} \otimes \mathbb{1}_B \, \sum_k (X_k \otimes \mathbb{1}_B) \, \sum_{i,j} r_{ij} \, A_i \otimes B_j \, (X_k^\dagger \otimes \mathbb{1}_B) \right) \tag{2.48a}\\
&= \sum_{i,j,k} \, r_{ij} \, \mathrm{Tr}_A \left(M_{a|x} \otimes \mathbb{1}_B \, (X_k \otimes \mathbb{1}_B) \, A_i \otimes B_j \, (X_k^\dagger \otimes \mathbb{1}_B) \right) \tag{2.48b}\\
&= \sum_{i,j,k} \, r_{ij} \, \mathrm{Tr}_A \left(M_{a|x} \, X_k \, A_i \, X_k^\dagger \otimes B_j \right) \tag{2.48c}\\
&= \sum_{i,j,k} \, r_{ij} \, \mathrm{Tr} \left(M_{a|x} \, X_k \, A_i \, X_k^\dagger \right) B_j \tag{2.48d}\\
&= \sum_{i,j,k} \, r_{ij} \, \mathrm{Tr} \left(X_k^\dagger \, M_{a|x} \, X_k \, A_i \, \right) B_j \tag{2.48e}\\
&= \sum_{i,j,k} \, r_{ij} \, \mathrm{Tr}_A \left(X_k^\dagger \, M_{a|x} \, X_k \, A_i \, \otimes B_j \right) \tag{2.48f}\\
&= \mathrm{Tr}_A \left((X_k^\dagger \otimes \mathbb{1}_B) \, M_{a|x} \otimes \mathbb{1}_B \, (X_k \otimes \mathbb{1}_B) \, \sum_{i,j,k} \, r_{ij} \, A_i \otimes B_j \right) \tag{2.48g}\\
&= \mathrm{Tr}_A \left((\Lambda^\eta \otimes \mathbb{1})^\dagger (M_{a|x} \otimes \mathbb{1}_B) \, \rho_{AB} \right) \tag{2.48h}\\
&= \mathrm{Tr}_A \left(\Lambda^{\eta\dagger}(M_{a|x}) \otimes \mathbb{1}_B \, \rho_{AB} \right), \tag{2.48i}
\end{align}$$

where in eq. (2.48e) the cyclic property of the trace was used and we define

$$\Lambda^{\eta\dagger}(A) = \sum_k X_k^\dagger A X_k \tag{2.49}$$

to be the adjoint map of the depolarizing channel.

For the partial trace, the cyclic property does not hold in general. In this particular case, however, it was possible to show that

$$\mathrm{Tr}_A \left(M_{a|x} \otimes \mathbb{1}_B \, (\Lambda^\eta \otimes \mathbb{1})(\rho_{AB}) \right) = \mathrm{Tr}_A \left(\Lambda^{\eta\dagger}(M_{a|x}) \otimes \mathbb{1}_B \, \rho_{AB} \right) \tag{2.50}$$

since we are dealing with operators of the form $A \otimes \mathbb{1}$, $A \in \mathcal{L}(\mathcal{H}_A)$. It is true that eq. (2.50) holds for any CPTP map, not only for the depolarizing channel.

However, the depolarizing channel in particular is also a self-adjoint map, $\Lambda^{\dagger\,\eta} = \Lambda^{\eta}$, *i.e.*,

$$\begin{aligned}
\mathrm{Tr}\left(A\ \Lambda^{\eta}(B)\right) &= \mathrm{Tr}\left[A\ \left(\eta B + (1-\eta)\mathrm{Tr}(B)\frac{\mathbb{1}}{d}\right)\right] &\text{(2.51a)}\\
&= \mathrm{Tr}(\eta AB) + \mathrm{Tr}\left[(1-\eta)\frac{1}{d}\,A\,\mathrm{Tr}(B)\right] &\text{(2.51b)}\\
&= \mathrm{Tr}(\eta AB) + (1-\eta)\frac{1}{d}\,\mathrm{Tr}(A)\,\mathrm{Tr}(B) &\text{(2.51c)}\\
&= \mathrm{Tr}(\eta AB) + \mathrm{Tr}\left[(1-\eta)\frac{1}{d}\,\mathrm{Tr}(A)\,B\right] &\text{(2.51d)}\\
&= \mathrm{Tr}\left[\left(\eta A + (1-\eta)\mathrm{Tr}(A)\frac{\mathbb{1}}{d}\right)\ B\right] &\text{(2.51e)}\\
&= \mathrm{Tr}\left(\Lambda^{\eta}(A)\ B\right). &\text{(2.51f)}
\end{aligned}$$

Thus, it is true that

$$\mathrm{Tr}_A\left(\Lambda^{\dagger\,\eta}(M_{a|x})\otimes\mathbb{1}_B\ \rho_{AB}\right) = \mathrm{Tr}_A\left(\Lambda^{\eta}(M_{a|x})\otimes\mathbb{1}_B\ \rho_{AB}\right) \tag{2.52}$$

and therefore,

$$\sigma_{a|x}^{\eta_A} = \mathrm{Tr}_A\left(M_{a|x}\otimes\mathbb{1}_B\ (\Lambda^{\eta}\otimes\mathbb{1})(\rho_{AB})\right) = \mathrm{Tr}_A\left(\Lambda^{\eta}(M_{a|x})\otimes\mathbb{1}_B\ \rho_{AB}\right). \tag{2.53}$$

Following from the definition on eq. (2.37),

$$\Lambda^{\eta}(M_{a|x}) = \eta M_{a|x} + (1-\eta)\mathrm{Tr}(M_{a|x})\frac{\mathbb{1}_{d_A}}{d_A}. \tag{2.54}$$

This is the depolarization of the measurements performed by Alice. We have shown that we can equally represent the action of the white noise on Alice's experiment by considering it to be affecting her part of the quantum state or equivalently her measurement elements. The assemblages resultant from both cases are equivalent.

Qubit measurements can also be written in the Pauli basis, with a 4-vector $\vec{u} = (\alpha, \vec{v})$, where α is a real positive number and $\vec{v}$ a real 3-vector that will also be called Bloch vector. Let $\{M_a\}$ be an a-outcome qubit POVM. We write

$$M_a = \frac{1}{2}\left(\alpha_a\,\mathbb{1} + \vec{v}_a\cdot\vec{\sigma}\right), \tag{2.55}$$

where, in order for $\{M_a\}$ to be a valid POVM, these three conditions must be satisfied:

(i) $\sum_a \alpha_a = 2$,

(ii) $\sum_a \vec{v}_a = 0$,

(iii) $\alpha_a \geq \|\vec{v}_a\|\,, \quad \forall\, a$.

In the particular case of projective measurements, the Bloch vector will always be a unit vector and $\alpha_a = 1$. Furthermore, since there are only two elements to a qubit projective measurement, the second Bloch vector will always be determined by the first as $\vec{v}_1 = -\vec{v}_2$. This means there are only two degrees of freedom to a projective measurement and it can be parametrized by two angles in the Bloch sphere.

Nevertheless, for general POVMs, the representation of qubit measurements on the Bloch sphere is far trickier since now there are four degrees of freedom to each POVM element and the last POVM element will be uniquely defined by the others in order to satisfy conditions in items (i) to (iii). In the course of this dissertation, we will represent POVM elements by its Bloch vectors on the Bloch sphere, but it is crucial that the fact that we are missing a dimension on this representation is not forgotten in order to avoid misleading interpretations.

Similarly to the effect of the depolarizing map on a qubit state, its effect on a qubit POVM is the equal shrinking of the Bloch vectors of all of its elements.

$$\Lambda^\eta(M_a) = \eta\,\frac{1}{2}\left(\alpha_a\,\mathbb{1} + \vec{v}_a\cdot\vec{\sigma}\right) + (1-\eta)\,\alpha_a\frac{\mathbb{1}}{2} \tag{2.56a}$$

$$= \frac{1}{2}\left(\alpha_a\,\mathbb{1} + \eta\,\vec{v}_a\cdot\vec{\sigma}\right), \tag{2.56b}$$

since $\mathrm{Tr}(M_a) = \alpha_a$. Notice that the parameter α_a is left unchanged. The effect of the depolarizing channel on POVMs can be seen as $(\alpha_a, \vec{v}_a) \to (\alpha_a, \eta\,\vec{v}_a)$.

For example, suppose one wants to represent a 3-outcome qubit POVM $\{M_a\}_{a=1}^3$ whose elements are given by $\alpha_1 = \alpha_2 = \alpha_3 = \frac{2}{3}$ and $\vec{v}_1 = (0, 0, \frac{2}{3})$, $\vec{v}_2 = (\frac{1}{\sqrt{3}}, 0, -\frac{1}{3})$, and $\vec{v}_3 = (-\frac{1}{\sqrt{3}}, 0, -\frac{1}{3})$. This POVM, which we will call regular trine POVM, can be represented in the Bloch sphere according to fig. 2.3a and the effect of the depolarizing channel $\Lambda^\eta(M_a)$ is the shrinking of the Bloch vectors represented on fig. 2.3b.

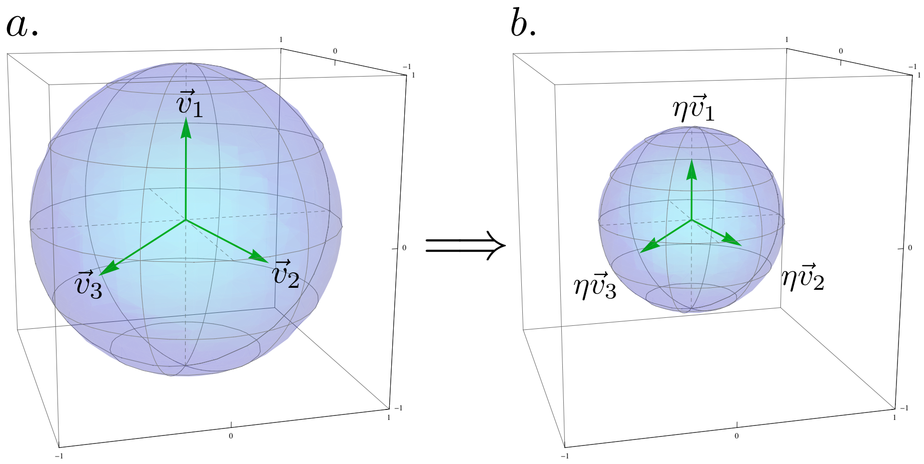

Figure 2.3: The shrinking of the Bloch sphere by action of the depolarizing channel $\Lambda^{\eta}(M_a)$ on qubit POVM elements.

Now let us analyze an assemblage resulting from local measurements on a bipartite state in the case where Bob's subsystems have been through a depolarizing channel:

$$\sigma_{a|x}^{\eta_B} = \mathrm{Tr}_A\left(M_{a|x} \otimes \mathbb{1}_B \ (\mathbb{1} \otimes \Lambda^{\eta})(\rho_{AB})\right). \tag{2.57}$$

Plugging in eq. (2.43b) for $(\mathbb{1} \otimes \Lambda^{\eta})(\rho_{AB})$ one arrives at

$$\sigma_{a|x}^{\eta_B} = \eta\,\mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}_B \ \rho_{AB}) + (1-\eta)\,\mathrm{Tr}_A\left(M_{a|x}\rho_A \otimes \frac{\mathbb{1}_B}{d_B}\right) \tag{2.58a}$$

$$= \eta\,\sigma_{a|x} + (1-\eta)\,\mathrm{Tr}(M_{a|x}\rho_A)\frac{\mathbb{1}_B}{d_B} \tag{2.58b}$$

$$= \eta\,\sigma_{a|x} + (1-\eta)\,p(a|x)\frac{\mathbb{1}_B}{d_B}. \tag{2.58c}$$

This is exactly the result one gets when applying the depolarizing channel directly on the assemblage elements:

$$\sigma_{a|x} \mapsto \Lambda^{\eta}(\sigma_{a|x}) = \eta\,\sigma_{a|x} + (1-\eta)\,\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}_B}{d_B}, \tag{2.59}$$

where $\mathrm{Tr}(\sigma_{a|x}) = \mathrm{Tr}(p(a|x)\rho_{a|x}) = p(a|x)$. Notice that this is equivalent to eq. (2.33) in the definition of the steering quantifier for assemblages white noise robustness. The white noise robustness of an assemblage $\{\sigma_{a|x}\}$ can be seen as the minimum value of η for which the assemblage $\{\Lambda^{\eta}(\sigma_{a|x})\}$ is unsteerable.

We have shown that the effect of white noise acting on Bob's experiment is equivalently represented by its action on Bob's state or assemblage:

$$\sigma_{a|x}^{\eta_B} = \Lambda^{\eta}(\sigma_{a|x}) = \mathrm{Tr}_A\left(M_{a|x} \otimes \mathbb{1}_B \ (\mathbb{1} \otimes \Lambda^{\eta})(\rho_{AB})\right). \tag{2.60}$$

Furthermore, if the global quantum state at hand is the pure maximally entangled state $\rho_{AB} = |\psi\rangle\langle\psi|$, such that $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$, it is possible to show that

$$\sigma_{a|x}^{\eta_B} = \sigma_{a|x}^{\eta_A}. \tag{2.61}$$

That is,

$$\Lambda^{\eta}(\sigma_{a|x}) = \mathrm{Tr}_A\left(M_{a|x} \otimes \mathbb{1}_B \ (\mathbb{1} \otimes \Lambda^{\eta})|\psi\rangle\langle\psi|\right) \tag{2.62a}$$

$$= \mathrm{Tr}_A\left(M_{a|x} \otimes \mathbb{1}_B \ (\Lambda^{\eta} \otimes \mathbb{1})|\psi\rangle\langle\psi|\right) \tag{2.62b}$$

$$= \mathrm{Tr}_A\left(\Lambda^{\eta}(M_{a|x}) \otimes \mathbb{1}_B \ |\psi\rangle\langle\psi|\right). \tag{2.62c}$$

The equivalence between eq. (2.62a) and eq. (2.62b) is a consequence of the fact that for this quantum state $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \frac{\mathbb{1}}{d} = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \rho_B$ and consequently

$$(\mathbb{1} \otimes \Lambda^{\eta})|\psi\rangle\langle\psi| = \eta|\psi\rangle\langle\psi| + (1-\eta)\frac{\mathbb{1}_A \otimes \mathbb{1}_B}{d^2} = (\Lambda^{\eta} \otimes \mathbb{1})|\psi\rangle\langle\psi|. \tag{2.63}$$

It follows from this fact that $(\Lambda^{\eta})^T = \Lambda^{\eta}$ since, as it was shown on eq. (2.21), for these quantum states it holds that $(\mathbb{1} \otimes M)|\psi\rangle = (M^T \otimes \mathbb{1})|\psi\rangle$.

2.3.4 Geometric representation

To compare the three quantifiers presented on this chapter, let us say there is an assemblage $\{\sigma_{a|x}\}$ whose steerability one wishes to quantify. In the case of the steering weight, it is necessary to optimize over all decompositions $\{\pi_{a|x}^{\text{uns}}\}, \{\pi_{a|x}^{\text{ste}}\}$ to calculate sw. In the case of the generalized robustness, it is necessary to optimize only over $\{\pi_{a|x}\}$ to calculate gr. Finally, in the case of white noise robustness, since the "disturbance" is fixed, it is only necessary to calculate the critical value of η for which the transition steerable/unsteerable happens in order to find out wnr.

However, this does *not* mean that the task of calculating one quantifier is more difficult than the other: they are all solved by semidefinite programming. This means

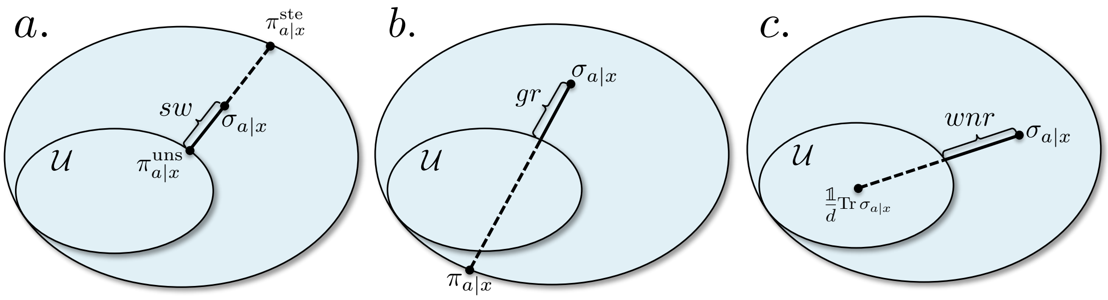

Figure 2.4: Geometric representation of steering quantifiers: *a*. Steering weight (section 2.3.1). *b*. Generalized robustness of steering (section 2.3.2). *c*. White noise robustness of steering (section 2.3.3).

that these are efficiently solvable problems and that, for reasonably low dimension and low number of elements in the assemblage, they can be solved in commercial machines demanding practicable amounts of computational time.

These quantifiers can be interpreted as definitions of distances in the set of assemblages between the assemblage of interest and the set unsteerable assemblages. However, one must be cautious with this kind of interpretation since geometric representations can be misleading, specially when trying to represent a multi-dimensional vector space on the two-dimensional surface of a sheet of paper (or computer screen). Still, for the sake of didactics, we provide a representation of steering weight, generalized robustness of steering, and white noise robustness of steering on fig. 2.4, to hopefully help the reader better assimilate these concepts.

One important remark is that, in the same way as these quantifiers were defined for steering, quantifiers cab be defined for the incompatibility of quantum measurements. Some examples are the *incompatibility weight*, *incompatibility robustness*, and *incompatibility random robustness*, all analogous to their steering counterparts, defined on reference [61].

2.3.5 Quantifying steerability of quantum states

In the previous subsections, some definitions of quantifiers for the steerability of assemblages were presented. Now, we move on to the problem of defining quantifiers for the steerability of quantum states.

The steerability *that can be demonstrated from a quantum state* depends completely on which measurements are being performed by the steering party on her part of the quantum state. The steerability *of a quantum state* is defined as the maximum steerability that can be demonstrated from it, optimizing over all possible measurements that can be performed by the steering party. It is equivalent to optimizing over all possible assemblages that can be generated from local measurements on a fixed quantum state.

Let us define the quantifiers of the previous subsections to bipartite quantum states. The steering weight $SW(\rho_{AB})$ of a bipartite quantum state ρ_{AB} is given by

$$SW\left(\rho_{AB}\right) = \max_{\{M_{a|x}\}} sw, \tag{2.64}$$

where sw from eq. (2.30) is the steering weight of an assemblage generated from $\{M_{a|x}\}$ measurements on part A of ρ_{AB}.

The general robustness of steering $GR(\rho_{AB})$ of the bipartite state ρ_{AB} is analogously defined as

$$GR\left(\rho_{AB}\right) = \max_{\{M_{a|x}\}} gr, \tag{2.65}$$

with gr defined in eq. (2.32).

Analogously, the white noise robustness of steering $WNR(\rho_{AB})$ of the bipartite state ρ_{AB} is

$$WNR\left(\rho_{AB}\right) = \max_{\{M_{a|x}\}} wnr, \tag{2.66}$$

with wnr defined in eq. (2.34).

None of this quantifiers can be calculated by semidefinite programming. In fact, there does not exist an efficient way to determine the value of this quantifiers, numerical nor analytical. In chapter 3, we present four original methods that can be used to estimate the steerability of bipartite quantum states subjected to a finite number of measurements. In chapter 4, we present our results on the implementation of such methods and advances in the calculation of the steerability of quantum states in scenarios with finite and infinite number of measurements, using mainly the white noise robustness of steering as our quantifier.

2.4 Further reading

In recent years, quantum steering has caught the attention of the scientific community, both for its applications on information processing tasks as for its appeal to the foundations of quantum mechanics. A variety of interesting results were produced, however, not all of them will be discussed in this dissertation. For this reason, in the following we present a collection of references where some results of the theory of quantum steering and experimental applications can be found, for the benefit of the reader.

Steering is defined for multipartite systems as the phenomenon of genuine multipartite steering in reference [77]. Experimental verifications of this phenomenon are presented in references [78] and [79]. In reference [16], the concept of postquantum steering is defined. The phenomenon of genuine hidden steerability is presented on reference [55]. A steering inequality equivalent to the CHSH-Bell inequality [39] is derived on reference [43]. An analytical criterion for deciding if a Bell-diagonal state is steerable with two or three projective measurements is derived in reference [80]. A counterexample for the later disproved Peres conjecture was found by presenting steerable bound entangled states in reference [81].

A one-sided device-independent quantum cryptography protocol is reported on reference [8]. Quantum randomness extraction using steering tasks in semi-device-independent levels of device characterization is presented in reference [82]. In reference [83], the first loophole-free steering test was reported, using polarization-entangled pairs of photons shared by two distant parties.

CHAPTER 3

Semidefinite Programming Formulation of Steering Problems

Semidefinite programming (SDP) is a very powerful tool in the approach of problems in steering theory, such as steering detection, quantification, and applications. Not only is it an instrument of calculation, but the formulation of apparently difficult problems as an SDP can help on the understanding of such problems.

SDPs can be efficiently[1] solved by available open softwares [84, 85]. In this work, they are used as a way of achieving results that would not be possible by means of analytical calculations and, crucially, often enlightening the way to general analytical results.

In this chapter we present the modeling of important problems in steering theory as SDPs, including the construction of LHS models and steering inequalities, the calculation of steering quantifiers for assemblages, and the calculation of the unsteerable bound and maximal violation of steering inequalities.

In section 3.3 we approach the most important topic in this chapter: the description of four novel methods for estimating the steerability of quantum states subjected to a finite number of measurements. They were developed by the author of this text and

[1] Formally, efficient programs are those which can be solved in polynomial time, *i.e.*, the time required to solve them scales polynomially with the number of variables in the program. This is true for SDPs. However, we also used the term "efficient" in the broader sense of the word, meaning that the programs we call efficient can be solved with commercial machines in reasonable amounts of time.

collaborators[2] in the project that led to this dissertation. All methods are optimization techniques based on SDP. For the first two methods, we take known techniques and reformulate them in order to apply to the specific problem of upper bounding the steerability of quantum states. The second two methods were developed exclusively to approach the problem of lower bounding the steerability of quantum states subjected to a finite number of measurements and have never been reported before. All four methods were formulated in order to approach a specific problem in steering theory and were subsequently proved to be applicable and efficient in a variety of other situations.

Semidefinite programming and other convex optimization techniques were used as tools in this work and an investigation of the theory behind them is beyond the goals of this text. For an introduction to these subjects the reader is referred to the book by Boyd and Vandenberghe [86] and the lecture notes by Ben-Tal and Nemirovski [87].

3.1 Membership problem

Membership problems are the kind of problems that arise when it is necessary to determine whether or not a given point is a member of a certain set. The one that concerns us is the problem of deciding whether a certain assemblage is a member of the set of unsteerable assemblages.

One way of checking this is to construct an LHS model for the assemblage at hand. If an assemblage admits an LHS model, then it is a member of the set of unsteerable assemblages. However, since local hidden variables can assume infinitely many forms and infinitely many values, this is, in principle, a difficult problem.

Another way is to check for violation of steering inequalities. If an assemblage violates a steering inequality, then it is not a member of the set of unsteerable assemblages. Nevertheless, since the convex set of unsteerable assemblages is not a polytope, there are infinitely many inequalities to be tested. Thus, in principle, this is a difficult problem as well.

Fortunately, both these problems can be solved by semidefinite programming (SDP), as was pointed out by references [25, 26]. This is usually very good news; SDPs are widely

[2] Marco Túlio Quintino, Leonardo Guerini, Thiago Maciel, Daniel Cavalcanti, and Marcelo Terra Cunha.

regarded as easily solvable problems, although like any other computational problem, the amount of computational power required scales with the size of the input (in our case, dimension and number of elements in the assemblage).

For the remaining of this section we will follow the work of reference [10].

3.1.1 Constructing LHS models

The SDP used to construct an LHS model for an assemblage is a **feasibility problem**. It is the kind of optimization problem that does not have an objective function to be minimized or maximized. A feasibility problem will look for, among the possible values its variables can assume, a solution that satisfies the constraints of the problem. If the solution does not exist, the problem is **infeasible**.

A very important distinction is that when an SDP returns the result "infeasible" it does not mean that the program *has not found* a solution to the problem. It actually means that the solution *does not exist*. For instance, a heuristic method may not provide an optimal solution because it was not able to find one. SDPs work differently, they are guaranteed to find the optimal solution when it exists and if an SDP result says the problem is infeasible, it is guaranteed that there is no solution.

In the case of constructing LHS models, among the set of all assemblages, the SDP will look for one assemblage $\{\sigma_\lambda\}$ that is able to classically mimic the statistics of the given assemblage $\{\sigma_{a|x}\}$. If a solution is found (the SDP is feasible), the assemblage is unsteerable and will return an LHS model; if the solution does not exist (the SDP is infeasible), it will be known that the assemblage is steerable. This SDP takes the following form:

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ \text{find} \quad & \{\sigma_\lambda\} \\ \text{subject to} \quad & \sigma_{a|x} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\ & \sigma_\lambda \geq 0 \quad \forall\, \lambda, \end{aligned} \tag{3.1}$$

where $D_\lambda(a|x) = D(a|x,\lambda)$ are the deterministic probability distributions. The input of the SDP must be a valid assemblage. Since $x \in \{1,\ldots,I_A\}$ and $a \in \{1,\ldots,O_A\}$

are Alice's inputs and outcomes, there are $I_A O_A$ elements in the assemblage $\{\sigma_{a|x}\}$. All elements $\sigma_{a|x}$ must be positive semidefinite operators in a d_B-dimensional complex Hilbert space, where d_B is the dimension of Bob's subsystem. They must satisfy the nonsignaling condition, $\sum_a \sigma_{a|x} = \sum_a \sigma_{a|x'}, \forall\, x, x'$ and the normalization condition $\sum_a \mathrm{Tr}\sigma_{a|x} = 1, \forall\, x$. Accordingly, all elements of the LHS assemblage $\{\sigma_\lambda\}$, which are variables of the SDP, must be operators acting on the same space $\mathcal{H}^{d_B}$ as the input assemblage, and $\lambda \in \Lambda = \{1, \ldots, O_A^{I_A}\}$. In principle, Λ could assume infinitely many values, but since there are only $O_A^{I_A}$ deterministic probability distributions $D_\lambda(a|x)$ in a scenario where Alice performs I_A measurements with O_A possible outcomes each, it is enough to consider this discrete and finite values for Λ. This means there will be $O_A^{I_A}$ elements in the LHS assemblage[3].

The first constraint, which is in fact a collection of $I_A O_A$ constraints – one for each pair (a, x) – is the LHS model itself, guaranteed to be valid by the positive semidefinite condition imposed on $\{\sigma_\lambda\}$ by the second constraint, which is in fact a collection of $O_A^{I_A}$ constraints – one for each λ. Notice that normalization condition of the LHS assemblage $\{\sigma_\lambda\}$ need not be imposed since from the second set of constraints, $\sum_a \mathrm{Tr}\sigma_{a|x} = \sum_\lambda \sum_a D_\lambda(a|x)\mathrm{Tr}\sigma_\lambda = \sum_\lambda \mathrm{Tr}\sigma_\lambda$ will automatically equal 1 if the input $\{\sigma_{a|x}\}$ is a valid assemblage.

It is possible to turn this feasibility problem into an optimization problem, which can be computationally advantageous [10]. This can be done by relaxing one of the constraints, like for example, the positivity condition of the assemblage elements σ_λ. If we impose the constraint $\sigma_\lambda \geq \mu\mathbb{1}$ and ask the SDP to maximize the value of μ, which will be the objective function of the problem, there will always exist a value of μ for

[3]This also means that the number of parameters n in each SDP, given by $n = d_B^2 O_A^{I_A}$, scales exponentially on the number of measurements I_A. Hence, as already mentioned, one of the greatest computational difficulties in our problems is running SDPs for large number of measurements.

which the problem is feasible. Rewriting the SDP (3.1),

$$
\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\max_{\{\sigma_\lambda\},\,\mu} \quad & \mu \\
\text{s.t.} \quad & \sigma_{a|x} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\
& \sigma_\lambda \geq \mu\mathbb{1} \quad \forall\, \lambda.
\end{aligned}
\tag{3.2}
$$

If the solution provides $\mu \geq 0$, all σ_λ are positive semidefinite, so there exists an LHS model and the assemblage is unsteerable. For any $\mu < 0$, $\{\sigma_\lambda\}$ is not a valid assemblage, so there does not exist an LHS description for $\{\sigma_{a|x}\}$. Hence, the input assemblage is steerable.

3.1.2 Constructing steering inequalities

One aspect of semidefinite programming theory that will be vastly explored here is the duality theory. Interestingly enough, to every membership problem that is characterized by SDP and strictly feasible[4], which will determine if a certain point belongs or not to a certain set, there is a dual problem that will provide a witness to the points that do not belong to this set. Therefore, a primal problem like the SDP (3.2) that provides LHS models to unsteerable assemblages, has a dual problem associated to it that will provide steering inequalities that are violated by steerable assemblages.

Let us explicitly calculate the dual of problem (3.2). The first step is to write the Lagrangian of the problem. As highlighted on chapter 5 of reference [86], the Lagrangian of a problem takes into account the constraints of the problem and decreases the objective function of a minimization problem by subtracting the constraints that correspond to non-negative functions, weighted by the Lagrange multipliers that will be the variables of the dual problem.

There will be one Lagrange multiplier for each constraint in the primal SDP. If a constraint of the primal problem is a constraint on operators, the Lagrange multiplier will be an operator. Likewise, if the constraint is on a scalar object, the Lagrange multiplier will be a scalar.

[4]An SDP is strictly feasible if the region of points that satisfies all of its constraints has an interior point.

In order to calculate the Lagrangian, which is a scalar function, it is convenient to rewrite both constraints of problem (3.2), which are constraints on operators, in terms of constraints on scalar variables.

Let $\{F_{a|x}\}_{a,x}$ be a set of operators that are the dual variables (Lagrange multipliers) to each constraint in the first set of constraints. Since $\sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\sigma_\lambda = 0\,\forall\, a, x$, it is true that the first set of constraints is equivalent to

$$F_{a|x}(\sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\sigma_\lambda) = 0 \quad \forall\, a, x, F_{a|x}, \tag{3.3}$$

without any restrictions on $F_{a|x}$. Making use of the relation $\mathrm{Tr}(AB) = 0\,\forall\, A$ iff $B = 0$, it is possible to state that

$$\mathrm{Tr}[F_{a|x}(\sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\sigma_\lambda)] = 0 \quad \forall\, a, x, F_{a|x}. \tag{3.4}$$

This statement is equivalent to the first set of constraints and will be satisfied by all values of $\{\sigma_\lambda\}$ that are a solution of problem (3.2).

Next, let $\{B_\lambda\}_\lambda$ be a set of operators that are the dual variables to each constraint in the second set of constraints. Since $\sigma_\lambda - \mu\mathbb{1} \geq 0$, $\forall\,\lambda$, it is true that the first set of constraints is equivalent to

$$B_\lambda(\sigma_\lambda - \mu\mathbb{1}) \geq 0, \quad \forall\, B_\lambda \geq 0,\ \lambda. \tag{3.5}$$

Using the relation $\mathrm{Tr}(AB) \geq 0\,\forall\, A \geq 0$ iff $B \geq 0$, it is possible to see that

$$\mathrm{Tr}[B_\lambda(\sigma_\lambda - \mu\mathbb{1})] \geq 0, \quad \forall\, B_\lambda \geq 0,\ \lambda. \tag{3.6}$$

This statement is equivalent to the second set of constraints and will be satisfied by all values of $\{\mu, \{\sigma_\lambda\}\}$ that are a solution of problem (3.2).

It is also convenient that in this problem the maximization of μ is equivalent to the minimization of $-\mu$. Consequently, the Lagrangian L can be written as

$$L\left(\mu, \{\sigma_\lambda\}, \{F_{a|x}\}, \{B_\lambda\}\right) = -\mu - \sum_{a,x} \mathrm{Tr}\left[F_{a|x}\left(\sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\sigma_\lambda\right)\right] - \sum_\lambda \mathrm{Tr}\left[B_\lambda\left(\sigma_\lambda - \mu\mathbb{1}\right)\right]. \tag{3.7}$$

Notice that the value of the Lagrangian is necessarily equal to or less than the value of the objective function $-\mu$, since we are subtracting positive and null functions from it.

The dual function g is the minimum value, for each set of values for the multipliers, that the Lagrangian can assume over the primal variables. It is given by

$$g\left(\{F_{a|x}\},\{B_\lambda\}\right) = \inf_{\mu,\{\sigma_\lambda\}} L\left(\mu,\{\sigma_\lambda\},\{F_{a|x}\},\{B_\lambda\}\right) \tag{3.8a}$$

$$= -\sum_{a,x}\mathrm{Tr}(F_{a|x}\sigma_{a|x}) + \inf_{\mu}\left\{\left(-1+\sum_\lambda \mathrm{Tr}(B_\lambda)\right)\mu\right\} \tag{3.8b}$$

$$+ \inf_{\{\sigma_\lambda\}}\left\{\sum_\lambda \mathrm{Tr}\left(\sum_{a,x} F_{a|x}D_\lambda(a|x) - B_\lambda\right)\sigma_\lambda\right\}.$$

The dual function yields lower bounds on the optimal value of the objective function of the primal problem. The trivial lower bound is $g\left(\{F_{a|x}\},\{B_\lambda\}\right) = -\infty$. Non-trivial lower bounds occur when all three following conditions hold:

(i) $-1+\sum_\lambda \mathrm{Tr}(B_\lambda) = 0$,

(ii) $\sum_{a,x} F_{a|x}D_\lambda(a|x) - B_\lambda = 0,\ \forall\,\lambda$,

(iii) and $B_\lambda \geq 0\ \forall\,\lambda$.

Hence,

$$g\left(\{F_{a|x}\},\{B_\lambda\}\right) = \begin{cases} -\sum_{a,x}\mathrm{Tr}(F_{a|x}\sigma_{a|x}), & \text{if items (i) to (iii) hold,} \\ -\infty, & \text{otherwise.}\end{cases} \tag{3.9}$$

If one maximizes the solution of the dual function over the dual variables, the maximal value among all lower bounds provided by it constitutes the solution of the dual problem:

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ \max_{\{F_{a|x}\},\{B_\lambda\}} \quad & -\sum_{a,x}\mathrm{Tr}(F_{a|x}\sigma_{a|x}) \\ \text{s.t.} \quad & \sum_\lambda \mathrm{Tr}(B_\lambda) = 1, \\ & \sum_{a,x} F_{a|x}D_\lambda(a|x) = B_\lambda,\ \forall\,\lambda, \\ & B_\lambda \geq 0\ \forall\,\lambda. \end{aligned} \tag{3.10}$$

Eliminating the mute variables B_λ and substituting the maximization of the objective function $f = -\sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x})$ by the minimization of $-f$, the final form of the dual of problem (3.2) is

$$\begin{aligned} &\text{given} \quad && \{\sigma_{a|x}\} \\ &\min_{\{F_{a|x}\}} && \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) \\ &\text{s.t.} && \sum_{a,x} F_{a|x} D_\lambda(a|x) \geq 0, \ \forall\, \lambda \\ & && \mathrm{Tr} \sum_{a,x,\lambda} F_{a|x} D_\lambda(a|x) = 1, \end{aligned} \tag{3.11}$$

where $x \in \{1, \ldots, I_A\}$ and $a \in \{1, \ldots, O_A\}$ labels Alice's inputs and outputs and $\lambda \in \{1, \ldots, O_A^{I_A}\}$. The input $\{\sigma_{a|x}\}$ must be a valid assemblage. The variables $F_{a|x}$ are Hermitian operators acting on $\mathcal{H}^{d_B}$, a complex Hilbert space of the dimension of Bob's subsystem. $D_\lambda(a|x)$ are deterministic probability distributions.

The interpretation of this problem is the following. If the input assemblage is steerable, the solution of the variables $\{F_{a|x}\}$ is a set of Hermitian operators that are the coefficients of the steering inequality $\beta = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}) \geq \beta^{\mathrm{uns}}$. This inequality is satisfied by all unsteerable assemblages and violated by the input assemblage. The objective function of this dual problem, which will be minimized, is the right-hand side of the inequality. The first set of constraints is responsible for determining β^{uns}. If one multiplies the constraint by σ_λ, sums over λ, and takes the trace, it follows that: $\sum_{a,x} \mathrm{Tr}(F_{a|x} \sum_\lambda D_\lambda(a|x)\sigma_\lambda) \geq 0 \implies \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma^{\mathrm{uns}}_{a|x}) \geq 0$, implying that the minimal value for β achieved by an unsteerable assemblage is $\beta^{\mathrm{uns}} = 0$. The second constraint has the role of fixing a scale for the inequality operators.

3.2 Quantifying steering for assemblages via SDP

As mentioned in section 2.3, steering quantifiers for assemblages can be calculated by semidefinite programming. We will now present the SDP formulation of the quantifiers previously discussed.

We will begin with the **steering weight** (section 2.3.1). The steering weight of an assemblage $\{\sigma_{a|x}\}$ is the solution of the following optimization problem:

$$\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\max \quad & \mu \\
\text{s.t.} \quad & \sigma_{a|x} = \mu\, \pi^{\text{uns}}_{a|x} + (1-\mu)\, \pi^{\text{ste}}_{a|x} \quad \forall\, a, x \\
& \pi^{\text{uns}}_{a|x} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\
& \sum_a \pi^{\text{ste}}_{a|x} = \sum_a \pi^{\text{ste}}_{a|x'} \quad \forall\, x, x' \\
& \pi^{\text{ste}}_{a|x} \geq 0, \quad \operatorname{Tr} \sum_a \pi^{\text{ste}}_{a|x} = 1 \quad \forall\, a, x \\
& \sigma_\lambda \geq 0 \quad \forall\, \lambda,
\end{aligned} \tag{3.12}$$

where $x \in \{1, \ldots, I_A\}$ and $a \in \{1, \ldots, O_A\}$ labels Alice's inputs and outputs and $\lambda \in \{1, \ldots, O_A^{I_A}\}$. The first set of constraints is the decomposition of the input assemblage into unsteerable and steerable parts. The second set of constraints is the existence of an LHS model for the unsteerable part of the decomposition of $\sigma_{a|x}$. The third set of constraints is the nonsignaling condition for the steerable part of the decomposition, which is automatically satisfied by the unsteerable part by the definition of the LHS model. Finally, the remaining constraints guarantee positivity and normalization of the assemblages $\{\pi^{\text{ste}}_{a|x}\}$ and $\{\sigma_\lambda\}$, that are along with μ the variables of the optimization problem.

This optimization problem, however, is not in the form of an SDP, because it is not linear in the optimization variables μ, $\{\sigma_\lambda\}$ and $\{\pi^{\text{ste}}_{a|x}\}$. Nevertheless, with some substitutions we can rewrite this problem in the form of an SDP, as was showed by reference [26].

The first step is to eliminate the variable $\{\pi^{\text{ste}}_{a|x}\}$. Combining the first set of constraints with the positivity conditions on $\pi^{\text{ste}}_{a|x}$, we have

$$\pi^{\text{ste}}_{a|x} = \frac{1}{1-\mu}\left(\sigma_{a|x} - \mu \sum_\lambda D_\lambda(a|x)\sigma_\lambda\right) \geq 0 \quad \forall\, a, x. \tag{3.13}$$

As long as the input $\{\sigma_{a|x}\}$ is a valid assemblage, the nonsignaling condition for $\{\pi^{\text{ste}}_{a|x}\}$ will hold and so will the normalization. Thus, the only variables left in the problem are

μ and $\{\sigma_\lambda\}$. Defining the new variable $\tilde{\sigma}_\lambda = \mu\sigma_\lambda$, we have

$$\operatorname{Tr}\sum_\lambda \tilde{\sigma}_\lambda = \mu \operatorname{Tr}\sum_\lambda \sigma_\lambda = \mu \tag{3.14}$$

and

$$\sigma_{a|x} - \mu\sum_\lambda D_\lambda(a|x)\sigma_\lambda = \sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\tilde{\sigma}_\lambda \geq 0. \tag{3.15}$$

Now, only variables left in the problem are $\{\tilde{\sigma}_\lambda\}$. Relabeling $\tilde{\sigma}_\lambda \to \sigma_\lambda$, the calculation of the steering weight of an assemblage can be written as an SDP:

$$\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\mu^* = \max \quad & \operatorname{Tr}\sum_\lambda \sigma_\lambda \\
\text{s.t.} \quad & \sigma_{a|x} - \sum_\lambda D_\lambda(a|x)\sigma_\lambda \geq 0 \quad \forall\, a, x \\
& \sigma_\lambda \geq 0, \quad \forall\, \lambda,
\end{aligned} \tag{3.16}$$

and $sw = 1 - \mu^*$ is the steering weight of $\{\sigma_{a|x}\}$.

The dual of problem (3.16) can be calculated in the same way as it was done for problem (3.11). The result is the dual SDP

$$\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\mu^* = \min \quad & \sum_{a,x} \operatorname{Tr}(F_{a|x}\sigma_{a|x}) \\
\text{s.t.} \quad & \sum_{a,x} D_\lambda(a|x)F_{a|x} - \mathbb{1} \geq 0 \quad \forall\, \lambda \\
& F_{a|x} \geq 0, \quad \forall\, a, x.
\end{aligned} \tag{3.17}$$

This dual problem can be simply interpreted. The operators $\{F_{a|x}\}$ are the coefficients of a linear steering inequality. The objective function $\beta = \sum_{a,x}\operatorname{Tr}(F_{a|x}\sigma_{a|x})$ is the value obtained by the input assemblage. The first set of constrains defines the unsteerable bound of the inequality. By multiplying both sides by σ_λ, summing over λ and taking the trace, it follows that $\operatorname{Tr}\sum_{a,x,\lambda} F_{a|x}D_\lambda(a|x)\sigma_\lambda = \operatorname{Tr}\sum_{a,x} F_{a|x}\sigma^{\text{uns}}_{a|x} \geq 1 = \beta^{\text{uns}}$. Hence, the steering inequality is $\beta = \sum_{a,x}\operatorname{Tr}(F_{a|x}\sigma_{a|x}) \geq 1$. Any assemblage with $\beta < 1$ violates the inequality and is guaranteed to be steerable. The last set of constraints enforces the minimum value for β of any assemblage to be 0.

Since the primal problem is strictly feasible [26], primal and dual problems satisfy the condition of strong duality [86]. Therefore, the optimal value of the objective function of the dual problem equals $\mu^* = 1 - sw$ of the input assemblage. This means that $\beta = \mu^*$, which is expected since μ^* is a non-negative number, less than one for steerable assemblages, and equal to one for unsteerable assemblages.

Next, the calculation of the **generalized robustness of steering** (section 2.3.2) is formulated as an optimization problem, still not in the form of an SDP. First, we will rewrite eq. (2.31) as

$$\gamma_{a|x} = \frac{\sigma_{a|x} + t\pi_{a|x}}{1+t} \tag{3.18}$$

with the correspondence between p and t being $p = \frac{1}{1+t}$. The reason behind this reparametrization is that the numerical calculation of the parameter t has showed far less numerical instability than the calculation of parameter p. The generalized robustness of the input assemblage $\{\sigma_{a|x}\}$ is obtained from the solution of the optimization problem

$$\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\min \quad & t \\
\text{s.t.} \quad & \gamma_{a|x} = \frac{\sigma_{a|x} + t\pi_{a|x}}{1+t} \quad \forall\, a, x \\
& \gamma_{a|x} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\
& \sum_a \pi_{a|x} = \sum_a \pi_{a|x'} \quad \forall\, x, x' \\
& \pi_{a|x} \geq 0, \quad \mathrm{Tr} \sum_a \pi_{a|x} = 1 \quad \forall\, a, x \\
& \sigma_\lambda \geq 0 \quad \forall\, \lambda,
\end{aligned} \tag{3.19}$$

where $x \in \{1, \ldots, I_A\}$ and $a \in \{1, \ldots, O_A\}$ labels Alice's inputs and outputs and $\lambda \in \{1, \ldots, O_A^{I_A}\}$, $\{\gamma_{a|x}\}$ is an unsteerable assemblage, and $\{\pi_{a|x}\}$ is any valid assemblage.

Very similarly to how the SDP for the steering weight was constructed, let us turn this optimization problem into an SDP, following the demonstration in references [26] and [70]. We will begin by eliminating variables $\{\pi_{a|x}\}$. From the first, second and fourth constraints,

$$\pi_{a|x} = \frac{1}{t}\left((1+t)\sum_\lambda D_\lambda(a|x)\sigma_\lambda - \sigma_{a|x}\right) \geq 0. \tag{3.20}$$

By the same reasoning as for the steering weight, the third constraint will be automatically satisfied. Defining $\tilde{\sigma}_\lambda = (1+t)\sigma_\lambda$,

$$\operatorname{Tr}\sum_\lambda \tilde{\sigma}_\lambda = (1+t)\operatorname{Tr}\sum_\lambda \sigma_\lambda = 1+t. \tag{3.21}$$

Relabeling $\tilde{\sigma}_\lambda \to \sigma_\lambda$, we arrive at the SDP:

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ t^* = \min \quad & \operatorname{Tr}\sum_\lambda \sigma_\lambda - 1 \\ \text{s.t.} \quad & \sum_\lambda D_\lambda(a|x)\sigma_\lambda - \sigma_{a|x} \geq 0 \quad \forall\, a, x \\ & \sigma_\lambda \geq 0, \quad \forall\, \lambda, \end{aligned} \tag{3.22}$$

and the generalized steering robustness of $\{\sigma_{a|x}\}$ will be $gr = \frac{t^*}{1+t^*}$.

The dual of this problem is

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ t^* = \max \quad & \sum_{a,x} \operatorname{Tr}(F_{a|x}\sigma_{a|x}) - 1 \\ \text{s.t.} \quad & \mathbb{1} - \sum_{a,x} D_\lambda(a|x) F_{a|x} \geq 0 \quad \forall\, \lambda \\ & F_{a|x} \geq 0, \quad \forall\, a, x, \end{aligned} \tag{3.23}$$

and the solution defines the steering inequality $1 + t^* = \sum_{a,x} \operatorname{Tr}(F_{a|x}\sigma_{a|x}) \leq 1$.

Finally, we can also formulate the problem of calculating the **white noise robustness of steering** (section 2.3.3) of an assemblage in the form of an SDP:

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ \eta^* = \max \quad & \eta \\ \text{s.t.} \quad & \eta\, \sigma_{a|x} + (1-\eta)\operatorname{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\ & \sigma_\lambda \geq 0, \quad \forall\, \lambda. \end{aligned} \tag{3.24}$$

where $x \in \{1, \ldots, I_A\}$ and $a \in \{1, \ldots, O_A\}$ labels Alice's inputs and outputs and $\lambda \in \{1, \ldots, O_A^{I_A}\}$.

Notice that this is also a relaxation of the feasibility problem (3.1) that turns it into an optimization problem. In this case, instead of relaxing the positivity condition of the LHS assemblage $\{\sigma_\lambda\}$ as was done in problem (3.2), it is possible to mix the input assemblage $\{\sigma_{a|x}\}$ with white noise and ask what is the minimum amount of white noise (maximum η) necessary to turn the input assemblage into one that admits an LHS description. The white noise robustness of the input assemblage $\{\sigma_{a|x}\}$ will be $wnr = 1 - \eta^*$.

By changing the optimization parameters, this SDP can be written in two other forms that can be more or less convenient[5]. The first one is the substitution $\eta = 1 - \mu$, which will turn the SDP (3.24) into a minimization problem:

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ \mu^* = \min \quad & \mu \\ \text{s.t.} \quad & (1-\mu)\sigma_{a|x} + \mu \mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\ & \sigma_\lambda \geq 0, \quad \forall \lambda. \end{aligned} \tag{3.25}$$

The white noise robustness of the input assemblage will be $wnr = \mu^*$.

The dual problem of problem (3.25) is

$$\begin{aligned} \text{given} \quad & \{\sigma_{a|x}\} \\ \mu^* = \max \quad & \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) \\ \text{s.t.} \quad & 1 - \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) + \frac{1}{d}\sum_{a,x} \mathrm{Tr}(F_{a|x})\mathrm{Tr}(\sigma_{a|x}) = 0 \\ & \sum_{a,x} D_\lambda(a|x) F_{a|x} \leq 0 \quad \forall\, \lambda. \end{aligned} \tag{3.26}$$

Its solution provides a steering inequality with precisely $\beta = wnr$ of the input assemblage. The steering inequality defined by the solution of this SDP has unsteerable bound equal to zero. The inequality is $wnr = \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) \leq 0$ and will be violated by a steerable assemblage, which has $wnr > 0$.

The second other form of the SDP for the white noise robustness of steering is achieved by using the relation equivalent to eq. (2.33),

$$\gamma_{a|x} = \frac{\sigma_{a|x} + t\,\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d}}{1+t}, \tag{3.27}$$

[5]By convenient, we mean mostly numerically stable.

and the optimization problem

$$
\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
\min \quad & t \\
\text{s.t.} \quad & \sum_\lambda D_\lambda(a|x)\sigma_\lambda = \frac{\sigma_{a|x} + t\,\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d}}{1+t} \quad \forall\, a, x \\
& \sigma_\lambda \geq 0 \quad \forall\, \lambda,
\end{aligned}
\tag{3.28}
$$

which is not an SDP since it is not linear on the optimization variables t and $\{\sigma_\lambda\}$. Applying the substitution from eq. (3.21), one arrives at the SDP:

$$
\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
t^* = \min \quad & \mathrm{Tr}\sum_\lambda \sigma_\lambda - 1 \\
\text{s.t.} \quad & \sigma_{a|x} + \left(\mathrm{Tr}\sum_\lambda \sigma_\lambda - 1\right)\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\
& \sigma_\lambda \geq 0, \quad \forall\, \lambda.
\end{aligned}
\tag{3.29}
$$

In this case, the white noise robustness of the input assemblage is given by $wnr = \frac{t^*}{1+t^*}$. Finally, the dual to this problem is

$$
\begin{aligned}
\text{given} \quad & \{\sigma_{a|x}\} \\
t^* = \max \quad & \frac{1}{d}\sum_{a,x}\mathrm{Tr}(F_{a|x})\mathrm{Tr}(\sigma_{a|x}) - \sum_{a,x}\mathrm{Tr}(F_{a|x}\sigma_{a|x}) - 1 \\
\text{s.t.} \quad & \mathbb{1} - \frac{\mathbb{1}}{d}\sum_{a,x}\mathrm{Tr}(F_{a|x})\mathrm{Tr}(\sigma_{a|x}) + \sum_{a,x} D_\lambda(a|x)F_{a|x} \geq 0 \quad \forall\, \lambda.
\end{aligned}
\tag{3.30}
$$

The primal and dual problems (3.25), (3.26) and (3.29), (3.30) will be vastly used in chapter 4.

Other two interesting problems that can be solved by SDPs are, for a fixed steering inequality, the calculation of its unsteerable bound and its maximum violation.

The first one is: given a set of coefficients $\{F_{a|x}\}$, find the maximum value of

$\beta = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma^{\mathrm{uns}}_{a|x})$ that can be obtained by an unsteerable assemblage

$$\begin{aligned}
&\text{given} && \{F_{a|x}\} \\
\beta^{\mathrm{uns}} = &\max && \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma^{\mathrm{uns}}_{a|x}) \\
&\text{s.t.} && \sigma^{\mathrm{uns}}_{a|x} = \sum_{\lambda} D_{\lambda}(a|x)\sigma_{\lambda} \quad \forall\, a, x \\
& && \sigma_{\lambda} \geq 0 \quad \forall\, \lambda, \qquad \mathrm{Tr} \sum_{\lambda} \sigma_{\lambda} = 1.
\end{aligned} \tag{3.31}$$

This SDP is useful when dealing with inequalities whose unsteerable bounds are not obvious like in the case of inequalities that come from dual problems of some steering quantifiers.

The second one is: given a set of coefficients $\{F_{a|x}\}$, find the maximum value of $\beta = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x})$ that can be obtained by a valid assemblage:

$$\begin{aligned}
&\text{given} && \{F_{a|x}\} \\
\beta^{\max} = &\max && \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}) \\
&\text{s.t.} && \sum_{a} \sigma_{a|x} = \sum_{a} \sigma_{a|x'} \quad \forall\, x, x' \\
& && \sigma_{a|x} \geq 0 \quad \forall\, a, x, \qquad \mathrm{Tr} \sum_{a} \sigma_{a|x} = 1, \quad \forall\, x.
\end{aligned} \tag{3.32}$$

The solution will be the value of the maximal violation of the inequality and the characterization of the assemblage that maximally violates it.

3.3 Steering quantum states: measurement optimization

The task of certifying the steerability of a quantum state is not an easy one. While to certify the steerability of an assemblage one can run an SDP, to certify the steerability of a quantum state it is necessary to check whether or not all assemblages that can be generated by performing local measurements on the given quantum state are steerable. Even for a fixed number of measurements, for example two measurements, that are an infinite amount of different sets of measurements that can be constructed, by arranging the two measurements in different ways. So even in the case of a finite number of measurements, there are infinitely many assemblages to be tested.

The problem of certifying that a quantum state is steerable for N measurements is the problem of finding one set of N measurements that, when performed on the quantum state, will demonstrate steering. But some sets of measurements can demonstrate *more* steering than others, for some chosen quantifier of steerability. Hence, it would be necessary to perform an optimization over all possible sets of N measurements to find the optimal one.

The problem of certifying that a quantum state is unsteerable for N measurements is the problem of guaranteeing that there does not exist a set of N measurements that could demonstrate steering when applied to the quantum state. This problem can be even more difficult because it is necessary to guarantee that an infinite amount of assemblages admit an LHS model.

By choosing a steering quantifier, like the white noise robustness, one can calculate upper bounds for the parameter η for which the quantum state of interest is certified to be steerable for N measurements and lower bounds for η for which the quantum state is certified to be unsteerable for N measurements. This is the goal of the methods we developed and present in the following.

But before starting the description of the methods, let us study the structure of the problem a little further. We may begin by closely examining SDP (3.24):

$$\begin{aligned}
&\text{given} \quad && \{\sigma_{a|x}\} \\
&\max_{\eta,\,\{\sigma_\lambda\}} && \eta \\
&\text{s.t.} && \eta\, \sigma_{a|x} + (1-\eta)\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall\, a, x \\
& && \sigma_\lambda \geq 0, \quad \forall\, \lambda.
\end{aligned} \tag{3.33}$$

This is the problem of calculating the shrinking factor of an assemblage, *i.e.*, the maximal value of η for which the noisy assemblage is unsteerable (see section 2.3.3). This problem is linear on the variables η and $\{\sigma_\lambda\}$ and can be solved by semidefinite programming. By solving this problem, what is effectively happening is an optimization over all possible valid assemblages $\{\sigma_\lambda\}$ in order to find one that satisfies the constraint of constructing an LHS model for the input assemblage and that maximizes the shrinking factor η.

Now, let us consider the problem of calculating the white noise robustness of the state ρ_{AB}, instead of the assemblage $\{\sigma_{a|x}\}$. In this case, it is necessary to optimize over all possible measurements that can be performed locally by the steering party on ρ_{AB} to find out what is the most robust assemblage that can be generated from it. Formulating this problem in the language of optimization problems, we have

$$\begin{aligned}
&\text{given} && \rho_{AB} \\
\min_{\{M_{a|x}\}} &\max_{\eta,\{\sigma_\lambda\}} && \eta \\
&\text{s.t.} && \sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, \rho_{AB}), \quad \forall\, a, x \\
& && \eta\sigma_{a|x} + (1-\eta)\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{d} = \sum_\lambda D_\lambda(a|x)\sigma_\lambda \quad \forall a, x \\
& && \sigma_\lambda \geq 0, \quad \forall\, \lambda \\
& && M_{a|x} \geq 0, \quad \forall\, a, x \qquad \sum_a M_{a|x} = \mathbb{1}, \quad \forall\, x.
\end{aligned} \tag{3.34}$$

Compared to the previous problem, this one has one more set of variables, $\{M_{a|x}\}$. These variables have other sets of constraints associated to it that guarantee that they correspond to valid quantum measurements. This kind of problem is called a **min-max problem**, since the optimal value of its objective function is a result of a maximization over some variables and a minimization over the other variables. This problem is not linear on its variables and it cannot be calculated by semidefinite programming. It is, in fact, a very hard problem to solve.

To better understand it, let us analyze an example. Let $\rho_{AB} = |\Psi^-\rangle\langle\Psi^-|$ be the singlet state shared by Alice and Bob. Alice will try to steer this state by performing any two projective qubit measurements (with two outcomes labeled by $+, -$) on her part. Since the singlet state is invariant by rotation, we can parametrize Alice's two projective measurements by the angle between their respective Bloch vectors. Any two projective measurements that have the same angle between them[6] will be equivalent in this case. Suppose Alice chooses her first measurement $\{M_{+|1}, M_{-|1}\}$ to be in the Z direction. The second measurement $\{M_{+|2}, M_{-|2}\}$ can be fixed on the xz plane and parametrized by

[6] Whenever expressions about the direction or angle between qubit measurements are used, they are referring to the direction or angle between their respective Bloch vectors. This terminology will be used often.

the angle θ with respect to the first measurement.

$$M_{\pm|1} = \frac{1}{2}\left(\mathbb{1} \pm \sigma_Z\right), \tag{3.35}$$

$$M_{\pm|2} = \frac{1}{2}\left(\mathbb{1} \pm \sin\theta\, \sigma_X \pm \cos\theta\, \sigma_Z\right). \tag{3.36}$$

For any fixed set of measurements, that is, for any fixed θ, the critical value of the shrinking factor of the assemblage generated by performing these measurements on one part of the singlet state can be calculated by the SDP (3.33). But to calculate the critical shrinking factor of the singlet state itself, it is necessary to optimize over the result of the SDP for all values of $\theta \in [0, \frac{\pi}{2}]$.

The plot in fig. 3.1 illustrates the optimization in the min-max problem (3.34). The blue curve is the result of the SDP (3.33) for fixed values of θ. All points below it correspond to unsteerable assemblages, generated from two measurements defined by θ on the singlet state with shrinking factor η. All points above it represent steerable assemblages. If one examines the vertical red line, it is possible to interpret it as the "work" of the maximization part of the problem: for a fixed set of two projective measurements defined by θ_0, the SDP will ride up the red line until it finds the maximal value η_0 for which the assemblage is unsteerable. The job of the minimization part of the problem is to ride down the blue curve until it finds the minimum value η^* that is solution of an SDP with input measurements defined by θ^*, represented by the green dot, effectively optimizing over SDP solutions.

The example presented is one of the few problems studied in this dissertation where the solution can be found precisely. Most of the time it is very computational demanding to find a candidate to a solution and almost always impossible to guarantee whether or not this candidate is in fact the solution of the problem.

In the remaining of this section, four original methods for estimating the steerability of quantum states when subjected to a *finite* number of measurements are presented. The first two, based on search algorithms and on the seesaw algorithm, are effective for calculating upper bounds to the solution, and are a reformulation of known algorithms to the problem of steering. The last two, one involving interior and the other exterior

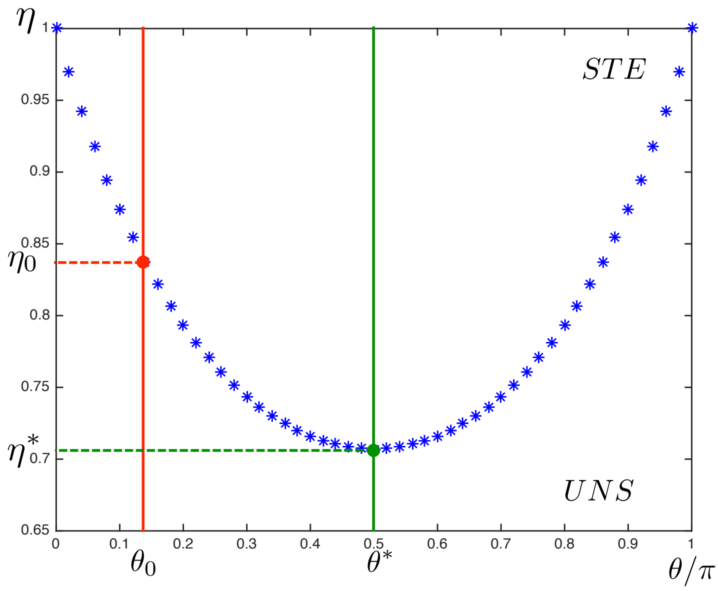

Figure 3.1: Plot of the min-max problem (3.34) solution for the shrinking factor of the singlet state subjected to two projective qubit measurements, parametrized by θ, performed locally by Alice. η^*, the critical shrinking factor, and θ^*, the parameter that defined the optimal set of measurements, are the solution of the problem. η_0 and θ_0 provide an upper bound to the solution, as does any other point in the blue curve.

approximation of convex sets by polytopes are effective in calculating lower bounds to the solution, and are novel methods.

3.3.1 Upper bounds: Search algorithms

From now on, instead of using the robustness of a quantum state to quantify its steerability, let us quantify the steerability of a quantum state by its critical shrinking factor, related to the white noise robustness by $\eta^*(\rho_{AB}) = 1 - WNR(\rho_{AB})$ (eq. (2.66)). The more steerable a quantum state is, the lower is its critical shrinking factor.

Since the calculation of the steerability of a fixed state and measurements is given by an SDP, our first method for calculating the steerability of a quantum state is to parametrize the quantum measurements allowed in a given scenario and, by varying these parameters, explore the solution of multiple SDPs to find the optimal value of problem (3.34).

Two important facts can be explored to facilitate this task. The first fact is that it is only necessary to optimize over extremal measurements. This is due to the fact that the shrinking factor depends linearly on the choice of the POVM, hence, by convexity, the optimal value will be obtained over extremal measurements. Extremal measurements are the ones that cannot be expressed as a convex combination of any other measurements in the set of all valid quantum measurements and that, in turn, all other measurements can be expressed as convex combinations of them [88]. The second fact is that for a quantum state of dimension d, there only exist extremal measurements of at most d^2 outcomes. Any measurement with more than d^2 outcomes can always be expressed as a convex combination of measurements with d^2 outcomes or less [88].

The heuristic method of optimization over measurements chosen for this work is the MATLAB function `fminsearch` [89], an unconstrained nonlinear multivariable optimization tool. Other possible choices are the MATLAB functions `fmincon` [90] and `fminunc` [91].

For the case of qubits we now present our parametrization. Consider a scenario where Alice is allowed to perform I_A projective measurements. All projective qubit measurements are extremal measurements. Each projective qubit measurement has a normalized Bloch vector associated to it (recall section 2.3.3). Thus, each measurement can be parametrized by two angles in the Bloch sphere, that will define the Bloch vector associated to one of its outcomes. The other one will be uniquely defined by the vector in the opposite direction. Explicitly,

$$M_{+|x} = \frac{1}{2}\left(\mathbb{1} + \hat{v}(\theta_x, \phi_x) \cdot \vec{\sigma}\right) \tag{3.37a}$$

$$M_{-|x} = \frac{1}{2}\left(\mathbb{1} - \hat{v}(\theta_x, \phi_x) \cdot \vec{\sigma}\right), \tag{3.37b}$$

where $\vec{\sigma}$ is the 3-vector of Pauli matrices and

$$\hat{v}(\theta_x, \phi_x) = (\sin\theta_x \cos\phi_x, \sin\theta_x \sin\phi_x, \cos\theta_x). \tag{3.38}$$

Each parameter lies in a range of $\theta_x \in [0, \pi]$ and $\phi_x \in [0, 2\pi]$. Yet, it is not necessary for them to be constrained since any angle outside of this range will only correspond to equivalent vectors that will yield redundant measurements. The total number of parameters in the optimization will be $2I_A$.

Now, for general POVMs the parametrization is not so direct. Let us begin by using the four parameters that define each POVM element:

$$M_{a|x} = \frac{1}{2}\left(\alpha_{a|x}\mathbb{1} + \vec{v}_{a|x}\cdot\vec{\sigma}\right). \tag{3.39}$$

The four parameters are $\alpha_{a|x}$ and the three coordinates in $\vec{v}_{a|x}$. Recall that they must satisfy these conditions:

(i) $\sum_a \alpha_{a|x} = 2 \quad \forall\, x,$

(ii) $\sum_a \vec{v}_{a|x} = 0 \quad \forall\, x,$

(iii) $\alpha_{a|x} \geq \left\|\vec{v}_{a|x}\right\|, \quad \forall\, a, x.$

The first two guarantee the POVM elements sum to identity and the third guarantees each POVM element is positive.

The problem is that there is no obvious range over which each of these parameters can be optimized in order to satisfy these conditions. Therefore, we take advantage of the fact that we only need to optimize over extremal measurements. One characteristic of extremal POVM elements is that they are proportional to rank-1 operators [88]. Thus, they can be rewritten as

$$M_{a|x} = \alpha_{a|x}\frac{1}{2}\left(\mathbb{1} + \hat{v}_{a|x}\cdot\vec{\sigma}\right), \tag{3.40}$$

where $\hat{v}_{a|x}$ is a unit vector that can also be parametrized by two angles $\theta_{a|x}$ and $\phi_{a|x}$. We will also take advantage of the fact that, for qubits ($d = 2$), we only have extremal measurements with up to 4 outcomes, so we will use $O_A = 4$. Any POVM with more than 4 outcomes can be written as a convex combination of 4-outcome POVMs and any POVM with less than 4 outcomes can be written as a 4-outcome POVM that has some elements equal to zero.

Now, the Bloch vector of the last POVM element is uniquely defined by the other 3 in order to satisfy the condition that the sum of the Bloch vectors of a POVM must equal zero. Hence, only $3(O_A - 1) = 9$ parameters are necessary for each POVM. The only condition that must be imposed during the optimization is that all $\alpha_{a|x}$ are non-negative and sum to 1.

Feeding this parametrization to `fminsearch` along with the SDP (3.33) and a random initial value for each parameter, the function will look for a set of parameters for which the solution of the SDP is as low as possible. This is an heuristic method so it is not guaranteed to find the actual solution of the problem (3.34), it will likely return a local minimum which is an upper bound to the critical shrinking factor of the input state. Running this nonlinear optimization a certain amount of times with different random initial points is a form of approximating the solution. However, the complexity and the demanded computational time increase rapidly with the dimension of the system, number of measurements, and number of outcomes of each measurement.

3.3.2 Upper bounds: The seesaw algorithm

The seesaw algorithm is an iterative method for solving some nonlinear optimization problems. Suppose one has an optimization problem of two variables that is bilinear, meaning it is linear on each of its variables individually but nonlinear as a whole. If the problems resulting from fixing one variable at a time are characterized by SDP, this is a kind of problem that can be solved by the seesaw method.

This algorithm is a sequence of two SDPs that are ran in a loop until convergence. In the first round of the loop, the first SDP can take a random input[7]. The output of the first SDP will be the input of the second SDP. In the next round of the loop, the output of the second SDP will be the input of the first SDP, and so on. The loop is terminated when some convergence condition between the outputs of each SDP is satisfied. The algorithm will to converge to a solution that is a bound for the solution of the original nonlinear problem[8]. A representation of the structure of a seesaw is found in algorithm 1.

Algorithm 1 . Structure of a seesaw algorithm.

```
x_1 = rand(n)                          ▷ n is the number of parameters in SDP_1
while <convergence condition> do
    x_2 = SDP_1(x_1)
    x_1 = SDP_2(x_2)
end while
```

[7]The initial input can also be an educated guess provided by the user.

[8]Under some special circumstances, it is possible to guarantee that the solution of the seesaw is exactly the solution of the original problem, but in general this is not the case.

In quantum information theory, the seesaw method finds many applications. In reference [81], a seesaw algorithm was used to find a counter-example for the Peres conjecture. In reference [10], the authors report a seesaw algorithm for measurement optimization.

We have developed a seesaw algorithm to approximate the solution of the nonlinear problem (3.34) of finding the critical shrinking factor of a quantum state.

Our seesaw iterates two SDPs. The first SDP is the dual of problem (3.33). For a given state, it will take a set of measurements as input and it will look, among all coefficients $\{F_{a|x}\}$ that satisfy the constraints of the problem, for the ones that define a specific steering inequality. This specific inequality is one that will be violated by the assemblage generated by the input measurements with $\sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) = \eta^*$, its shrinking factor. Then, the second SDP will take the coefficients of the steering inequality that were the solution of the previous SDP as input. It will look, among the set of quantum measurements, for the one that generates the assemblage that maximally violates the input inequality. The second SDP will output this set of measurements, which will become the input of the first SDP in the second round of the loop. The shrinking factor of the assemblage generated by this new set of measurements will necessarily be less or equal to the shrinking factor of the assemblage from the previous round.

Let us explicitly write the SDPs involved in our seesaw. The first one, which will begin taking a randomly chosen set of measurements as input, is

$$\begin{aligned}
\text{given} \quad & \{M_{a|x}\} \\
\min_{\{F_{a|x}\}} \quad & 1 - \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) \\
\text{s.t.} \quad & \sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, \rho_{AB}), \quad \forall\, a, x \\
& 1 - \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) + \frac{1}{d} \sum_{a,x} \mathrm{Tr}(F_{a|x})\mathrm{Tr}(\sigma_{a|x}) = 0 \\
& \sum_{a,x} D_\lambda(a|x) F_{a|x} \leq 0 \quad \forall\, \lambda.
\end{aligned} \tag{3.41a}$$

Either (3.26) or (3.30) are also appropriate. Of all three, our calculations have shown (3.30) to exhibit the least numerical instability.

The second SDP is simply

$$\begin{aligned} \text{given} \quad & \{F_{a|x}\} \\ \max_{\{M_{a|x}\}} \quad & \sum_{a,x} \mathrm{Tr}(F_{a|x}\sigma_{a|x}) \\ \text{s.t.} \quad & \sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, \rho_{AB}), \quad \forall\, a,x \\ & M_{a|x} \geq 0, \quad \forall\, a,x \qquad \sum_a M_{a|x} = \mathbb{1}, \quad \forall\, x. \end{aligned} \tag{3.41b}$$

Just like the numerical optimization from the previous section, this method can become impractical since the complexity of the problem increases with dimension, number of measurements and number of outcomes in each measurement. For a large number of parameters, this method will demand more computational time to find candidates to the solution. It will also result in local minima far more often.

Both our seesaw algorithm and our search algorithm are applicable to estimating the steerability of quantum states subjected to a finite number of general measurements. In this case, our calculations have shown the seesaw algorithm to be far more efficient[9] than the MATLAB function `fminsearch`. However, only the search method allows restrictions on the kind of measurements over which the program will optimize. For example, when using the search method, one could restrict the optimization to go over only projective measurements or over some specific POVMs. In these particular cases the search algorithm is the best option since, due to the formulation of SDP (3.41b), these restrictions are not allowed in the seesaw.

Our results on the implementation of these methods are presented in chapter 4.

3.3.3 Lower bounds: Outer polytope approximations

Now we begin the presentation of the methods that certify unsteerability of quantum states subjected to a finite number of measurements. To lower bound the critical shrinking factor of quantum states subjected to a finite number of measurements is a difficult problem. It is necessary to guarantee that, for all possible sets of N measurements performed on a fixed quantum state with a certain shrinking factor, none of them will

[9]Meaning that for same input, the seesaw algorithm was able to find the same solution that was found by the search algorithm but in amounts of time that are smaller than the required by the search algorithm in many orders of magnitude.

exhibit steerability. More than a problem of quantification of steering, this is a problem of guaranteeing the existence of an LHS model for all infinite assemblages that can be generated by locally performing a set of N measurements on a quantum state.

In references [92] and [93], the authors independently present a method for guaranteeing the existence of LHS models for quantum states subjected to an infinite amount of measurements, which provides a "trivial" bound to the case of finitely many measurements. These methods are very useful for constructing one-way steerable states. In reference [94], the authors use polytope approximation method to approach the problem of simulating general POVMs via projective measurements. Here, we use the technique of polytope approximation to lower bound the steerability of quantum states subjected to a *finite* number of measurements, *i.e.*, to calculate a value of η bellow which it is certified that a quantum state cannot exhibit steering when subjected to N measurements. Before the development of this method and the one presented on the next subsection, there were no techniques to approach this problem, up to the extent of our knowledge.

Consider the set $\mathcal{A}$ of all assemblages generated by I_A measurements of O_A outcomes each, performed locally on a bipartite quantum state ρ_{AB}. This set is convex but not a polytope, so there exists an infinite amount of extremal points in this set. The subset of unsteerable assemblages $\mathcal{U}$ is convex and also not a polytope.

As discussed earlier in this text, the effect of the depolarizing channel (eq. (2.37)) on the set of all assemblages is the shrinking of the whole set. If one applies the depolarizing channel on all steerable assemblages, there will be a value of η for which each of these assemblages will cross the boundary of the subset of unsteerable assemblages and become unsteerable themselves. Consequently, there will be a minimum value η^* for which the depolarizing channel causes all of the points that were outside the subset of unsteerable assemblages to be inside this subset. By the very definition of η^*, for any larger value there would be some assemblage which would be outside the set $\mathcal{U}$ and remain steerable.

This value is exactly the critical shrinking factor of ρ_{AB} when subjected to I_A measurements of O_A outcomes, the solution of problem (3.34). It is the minimum shrinking factor among those of all steerable assemblages generated by locally performing I_A measurements of O_A outcomes on ρ_{AB}. Notice it is not necessary to calculate the shrinking factor of *all* steerable assemblages. It is enough to calculate the shrinking factor of the extremal points of the set of all assemblages. The difficulty is that there is an infinite number of extremal points in this set, so this is not a practical solution. It would be a much simpler problem if the set of all assemblages were a polytope. In this case, it would only be necessary to calculate the shrinking factor of a finite number of points (the extremal points of the polytope) and find the minimum among them to obtain η^*.

In light of this discussion, the method we propose to estimate the critical shrinking factor of a quantum state is this: an approximation of the set of all assemblages generated by a fixed number of measurements on a given quantum state by an exterior polytope. This idea is pictorially represented in fig. 3.2*a*. If the set of all assemblages is contained by a polytope and this polytope is depolarized enough so that all of its extremal points become unsteerable assemblages, then it is true that all valid assemblages, which can all be expressed as convex combinations of the extremal points of the depolarized polytope, are also unsteerable. This is the main idea behind our method.

Some points inside the polytope, including its extremal points, will not correspond to valid assemblages. These are the points in the darker blue area in fig. 3.2*a*. In order to define them, some condition on the definition of assemblages must be relaxed. The condition we chose is the positivity condition. The extremal points of the polytope that approximates the set of all assemblages do not correspond to sets of positive semidefinite operators. However, applying the depolarizing channel to these points and guaranteeing that they are all depolarized enough to belong to the subset of unsteerable assemblages is enough to guarantee that the same has happened to all valid assemblages. This can be done by running one single SDP, the SDP (3.33), for each of the extremal points of the outer polytope.

Of course, the amount of white noise necessary to depolarize the polytope is larger than the necessary to depolarize the set of all assemblages. Hence, the minimum shrinking factor among the extremal points of the polytope is a lower bound for the actual critical shrinking factor of the quantum state at hand.

In the implementation of this method, the main obstacle is choosing the extremal points of the polytope and certifying that the set of assemblages is entirely contained by it. We present in details our implementation of the particular case of assemblages generated

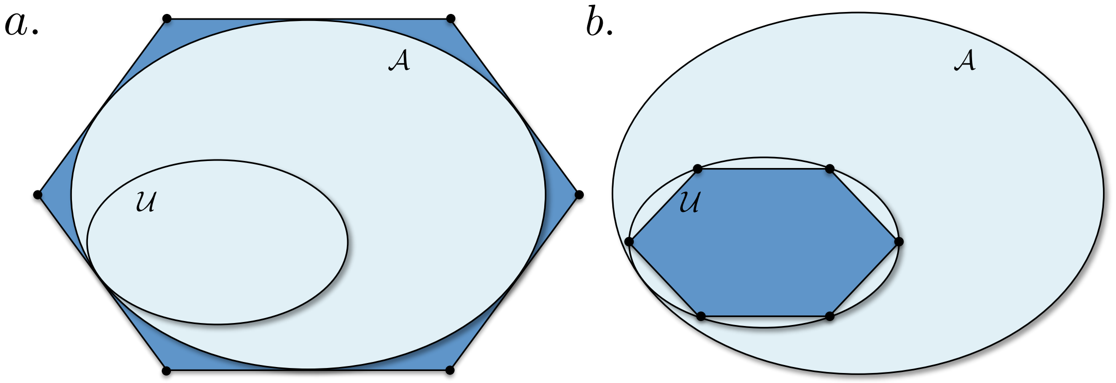

Figure 3.2: (a) Outer polytope approximation of the convex set $\mathcal{A}$ and (b) inner polytope approximation of the convex set $\mathcal{U}$.

by measurements on qubits to clarify the idea presented so far. This implementation is similar to the one proposed in reference [94], where the authors present a technique to approximate the set of valid POVMs by an exterior polytope.

The goal is to guarantee that, for a certain η, a certain quantum state is unsteerable for all possible sets of N measurements. Therefore, in the relaxation of the positivity condition of the elements of the assemblages, it is necessary to guarantee that the resulting objects are still related to the quantum state of interest somehow. Our approach is to build non-valid assemblages by performing non-valid measurements on valid quantum states, the quantum states we wish to study. That is, we relax the positivity condition of the elements of the measurements that are being performed locally on valid bipartite quantum states to generate non-valid assemblages. Hence, it is necessary to approximate the set of all *quantum measurements* by a polytope, instead of the set of all assemblages.

For bipartite states of the form $\rho_{AB} \in \mathcal{L}(\mathcal{H}^2 \otimes \mathcal{H}^d)$, where the steering party performs measurements on qubits, there is a straightforward way to generate the polytope that englobes the set of all POVMs.

Once again, we use the Bloch sphere representation of qubit measurements. Given a qubit measurement element $M_{a|x} = \frac{1}{2}(\alpha_{a|x}\mathbb{1} + \vec{v}_{a|x} \cdot \vec{\sigma})$, recall that the positivity condition is given by $\alpha_{a|x} \geq \left\| \vec{v}_{a|x} \right\|$. This means that the Bloch vector must be inside a sphere of radius $\alpha_{a|x}$ for $M_{a|x}$ to be positive semidefinite.

The trick is this: in order to approximate the set of quantum measurements with a

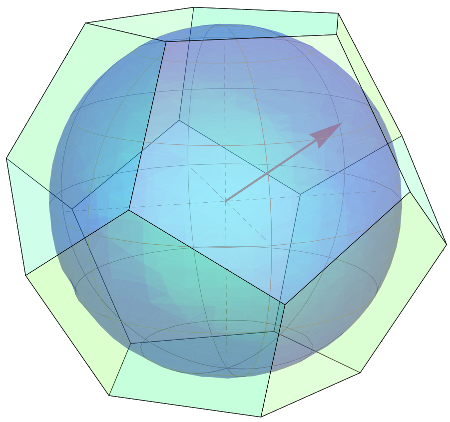

Figure 3.3: Approximation of the Bloch sphere by an external polytope.

polytope, we approximate the Bloch sphere by a polytope. The advantage is that we know the format of the Bloch sphere so it is easy to choose the extremal points of the polytope. One could choose, for example, a Platonic solid or some other notion of equally distributed points in a sphere, and set the inner radius of the solid to be equal to $\alpha_{a|x}$ (see fig. 3.3). Then, the extremal points will correspond to vectors with $\left\|\vec{v}_{a|x}\right\| > \alpha_{a|x}$, *i.e.*, non-positive measurement elements.

The extremal non-valid projective measurements are constructed by taking the vectors that correspond to each vertex of the polytope and their antipodals. The set of non-valid projective measurements that will be performed locally on the valid bipartite state is given by the cartesian product of all non-valid projective measurements that were constructed from the vertices of the polytope. Finally, the extremal non-valid assemblages are generated by performing the sets of non-valid extremal measurements on Alice's part of the quantum state. The lower bound for the critical shrinking factor of the quantum state with a finite number of measurements is calculated by running one SDP to each non-valid extremal assemblage and finding the minimum value of η among the solutions.

This approximation can be made to be as good as one wishes, up to available computational resources, by adding more extremal points to the outer polytope, each one demanding another run of the SDP. Still, this will only improve the bound. Only with an infinite amount of extremal points (and runs of the SDP) this method could yield the exact result.

Our results on the implementation of these methods are also presented in chapter 4.

3.3.4 Lower bounds: Inner polytope approximations

In the previous method, we overcame the problem that the set of all assemblages has an infinite amount of extremal points by approximating it with an exterior polytope. For this last method, we will overcome the problem that the set of unsteerable assemblages is not characterized by a finite amount of facets by approximating it with an interior polytope. This method is a generalization of the method presented in reference [93] and applied also in reference [95].

If there were a finite number of facets in the set of unsteerable assemblages, it would be characterized by a finite number of steering inequalities. By applying the depolarizing channel to the set of all assemblages with some fixed η, the whole set would be shrunk. Then it would be possible to test each of the inequalities defined by the facets of the polytope using SDP (3.32) to find out whether there exists some depolarized assemblage that violates it. If for some η none of the inequalities were violated by the shrunk set but an arbitrarily small increase in η would result in the violation of some inequality, it would be known that this value of η is the shrinking factor of the quantum state that generated the assemblages in the depolarized set.

Since in reality it is necessary to make an approximation of the set of unsteerable assemblages by an inner polytope, there are some subtleties to be considered. Take the inner polytope approximation represented on fig. 3.2*b*. The inequalities that are defined by the facets of the polytope are not usual steering inequalities. When all inequalities defined by the polytope are satisfied by the same assemblage, they guarantee that this assemblage is unsteerable. However, when violated these inequalities do not guarantee steerability, since there are unsteerable assemblages that are outside the polytope. Instead of being witnesses to steerability, like common steering inequalities, to satisfy all of these inequalities witnesses unsteerability.

Yet, given the facets of the polytope that internally approximates the set of unsteerable assemblages, the depolarizing and inequality testing procedure can be applied to estimate how much white noise is necessary in order to make all assemblages satisfy all inequalities and, hence, become unsteerable. This method only provides a lower bound because it is necessary more depolarization to make all assemblages satisfy all inequalities than it would be necessary to make all assemblages unsteerable. The bound can be refined as much as necessary by adding more facets to the polytope but never exactly calculated with an arbitrarily large yet finite number of facets.

The idea is similar to the method of external polytope approximation, except in this case the interest is on the facets of the polytope and not on the extremal points. There remains the difficulty in choosing the extremal points of the polytope, which in this case will be valid unsteerable assemblages, to construct the facets of the polytope. However, since the set of unsteerable assemblages will contain the polytope, it is not necessary to know its format in order to define the polytope. In fact, the convex hull of any number of extremal unsteerable assemblages would generate some polytope that can be used. Yet, in order to define the polytopes that better approximate the set of unsteerable assemblages and provide better bounds, it would be useful to know its format.

This method has not yet been computationally implemented by us but its conceptual value is important in the understanding of our problem. The inner and outer polytope approximations of convex sets can be used to solve similar problems involving any sort of convex sets, not only the ones discussed in this dissertation.

3.4 Further reading

The usefulness of SDPs in quantum information theory extrapolates its scope on steering problems. In this last section we list some of them, along with suiting references, as suggestions to the possibly interested reader.

The NPA hierarchy [96] is a hierarchy of SDPs that has been successfully used to approximate the set of quantum correlations that arise in a Bell scenario. The characterization of this set is one of the biggest open problems in quantum information. The Lovász number [97], which has many applications in graph theory, combinatorics, and quantum information, can be calculated by an SDP. It is a bound to the maximal quantum violation of Bell inequalities, a bound to contextually inequalities, a bound to zero-error channel capacity, and a bound to the independence number of a graph [30].

SDPs were recently showed to be useful tools in state and process tomography [98–100]. They were also used to find counter examples to the later disproved Peres conjecture [81]. In entanglement theory, the k-extension separability criterion can be tested via SDP [101] and the set of separable states can be approximated by the set of PPT states, which are characterized by SDP [102]. SDPs are also used in methods for proving the security of device-independent quantum key distribution protocols [103].

[redacted]

CHAPTER 4

Steering qubits with a Finite Number of Measurements

This chapter is devoted to reporting the novel results of this dissertation, product of the work of Jessica Bavaresco, Marco Túlio Quintino, Leonardo Guerini, Thiago Maciel, Daniel Cavalcanti, and Marcelo Terra Cunha.

In this last chapter we present our calculations on the quantification of the steerability that can be demonstrated by performing an arbitrary finite number of measurements on qubit states. They involve analytical calculations as well as the application of the numerical methods we developed and presented on the previous chapter.

The quantifier chosen to be the subject of our calculations is the white noise robustness of steering, defined in section 2.3.3. This quantifier has direct physical interpretation and high experimental interest, which motivated our choice. The parameter of interest is the shrinking factor η, related to the white noise robustness of steering by $\eta = 1 - wnr$.

The two-qubit states we explore are the two-qubit Werner states [27] (see section 1.7), which can be seen as simply the result of applying the depolarizing channel (eq. (2.37)) on the singlet state with different values of shrinking factor $\eta \in [0, 1]$.

They are defined by:

$$\rho_{AB} = \eta|\Psi^-\rangle\langle\Psi^-| + (1-\eta)\frac{\mathbb{1}}{4} \tag{4.1}$$

where

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{4.2}$$

is the singlet state.

Let us begin exploring these states by reviewing results in the literature on this subject.

For the case of projective measurements, it is known that Werner states with $\eta \leq \frac{1}{2}$ are unsteerable for any set of arbitrarily many projective measurements (including the infinite set of all projective measurements). This is a consequence of Werner's local model presented on reference [27].

For a finite number of measurements, the simplest scenario is the case of two measurements, since the case of only one measurement being performed by the steering party is trivial[1]. For the case of two projective measurement, it is known that with $\eta > \frac{1}{\sqrt{2}}$ Werner states violate the CHSH inequality when subjected to suitable sets of orthogonal projective measurements on each part [27]. Hence, in this range they are nonlocal and, consequently, steerable. But from the violation of the CHSH inequality, it is not possible to conclude anything about the steerability that can be demonstrated on Werner states with values of η slightly smaller than $\frac{1}{\sqrt{2}}$. There is more to be learned by exploring a result by Paul Busch [104] in the theory of joint measurability, enunciated below.

Theorem 5. *Let $\{A_\pm\}$ and $\{B_\pm\}$ be two dichotomic qubit measurements with respective Bloch vectors $\vec{a}$ and $\vec{b}$. Then, $\{A_\pm\}$ and $\{B_\pm\}$ are jointly measurable if, and only if,*

$$\left\|\vec{a} + \vec{b}\right\| + \left\|\vec{a} - \vec{b}\right\| \leq 2. \tag{4.3}$$

First it is necessary to translate our steering problem to a joint measurability one. As proved in theorem 3, a set of measurements will demonstrate steering if, and only if, it is not jointly measurable. So let us investigate the joint measurability of two noisy qubit projective measurements. Let these measurements be

$$M_{\pm|x} = \frac{1}{2}(\mathbb{1} \pm \eta \hat{v}_x \cdot \vec{\sigma}), \quad x \in \{0, 1\} \tag{4.4}$$

[1] For the case of only one measurement per party, the resulting statistics can always be classically simulated and described by LHS/LHV models. Hence, steering or nonlocality can never be demonstrated in this case.

where the Bloch vectors $\hat{v}_0, \hat{v}_1$ are two unit vectors at the same origin separated by an angle θ. Without lost of generality we can choose $\hat{v}_0 = (0, 0, 1)$ and $\hat{v}_1 = (\sin\theta, 0, \cos\theta)$. It follows that

$$\|\eta\hat{v}_0 + \eta\hat{v}_1\| + \|\eta\hat{v}_0 - \eta\hat{v}_1\| = \eta\left(\sqrt{2(1+\cos\theta)} + \sqrt{2(1-\cos\theta)}\right). \tag{4.5}$$

According to theorem 5, they will be jointly measurable if, and only if

$$\eta\left(\sqrt{2(1+\cos\theta)} + \sqrt{2(1-\cos\theta)}\right) \leq 2. \tag{4.6}$$

Hence,

$$\eta \leq \frac{1}{\sqrt{\frac{1+\cos\theta}{2}} + \sqrt{\frac{1-\cos\theta}{2}}} \tag{4.7a}$$

$$\eta \leq \frac{1}{\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})} \tag{4.7b}$$

$$\eta \leq \frac{1}{\sqrt{1+\sin\theta}}. \tag{4.7c}$$

The minimum value of the right hand side of the above inequality is at $\theta = \frac{\pi}{2}$ implying that any set of two noisy qubit projective measurements is jointly measurable if $\eta \leq \frac{1}{\sqrt{2}}$. Consequently, any set of two projective measurements acting locally on the singlet state (or any other two-qubit state) will not demonstrate steering for this range of η. This joint measurability result, along with the CHSH inequality violation, proves that $\frac{1}{\sqrt{2}}$ is the critical shrinking factor of the singlet state for two projective measurements. Notice also that the ineq. (4.7c) coincides exactly with the blue curve in fig. 3.1, result of running the SDP (3.33) for different values of θ.

This is one of the few cases that can be analytically solved to some extent. There is not yet a final answer for the steerability of the singlet state with two general POVMs and whether or not the use of POVMs could decrease the shrinking factor of this state, demonstrating higher steerability than with projective measurements.

Other results found in the literature include: for the case of three orthogonal projective measurements, reference [41] reports that Werner states are steerable with $\eta > \frac{1}{\sqrt{3}}$. For the cases of 4, 6, and 10 projective measurements whose Bloch vectors correspond to the vertices of platonic solids, calculations and experiments are reported on reference [105].

These are results for fixed sets of measurement. Very little is known about the optimal case in each scenario, *i.e.*, what is the critical value of η for which Werner states are steerable when optimizing over all possible sets of a fixed number of measurements.

The main goal of this work is twofold:

(i) Characterize the steerability of the two-qubit Werner states subjected to a finite number of measurements, using the parameter η.

(ii) Discover what are the optimal sets of measurements that demonstrate maximal steerability on each scenario.

The optimal set of measurements is also the most incompatible set of measurements and its incompatibility can be used as a resource as well [60, 61]. Ultimately, we are interested in exploring the relevance of general POVMs over projective measurements. The question that interests us the most is whether or not there exist circumstances under which general POVMs can be used to demonstrate steering while projective measurements cannot.

In the following, we present our original results, devided in three main groups: steerability of the qubit Werner states with the family of equatorial (planar) qubit projective measurements, the ones restricted to the equatorial plane of the Bloch sphere; steerability of the qubit Werner states with the family of general projective qubit measurements, the ones all around the Bloch sphere; and, finally, the relevance of general POVMs to the steerability of the Werner state.

In the last section we show preliminary results of the application of our methods to other problems, showing that the methods we developed are applicable not only to the study of the Werner state, but are extendable and efficient in the study of other families of quantum states, of higher dimension and different properties, using different steering quantifiers. We also discuss how our methods can be adapted to the study of Bell nonlocality.

4.1 Planar projective measurements

One interesting family of qubit measurements is the family of planar measurements. They are the ones, either projective or more general POVMs, whose Bloch vectors are restricted to the same plane, for example, the equator of the Bloch sphere or the xz plane. There are three main reasons to study this kind of measurements. The first one is for its fundamental value: planar measurements correspond to real vector space operators. The study of real vector space quantum mechanics has recently had attention devoted to it, as new phenomena arise under these circumtances [106, 107]. The second one is experimental: real vector space measurement and states have an experimental appeal since under some circumstances their implementation is simpler than for general measurements [108]. The last one is numerical: to optimize over sets of planar projective measurements is less computationally demanding than to optimize over general projective measurements. In addition, one can hope that learning about the properties of this simpler case can bring insights on how to solve more difficult and general problems.

We begin by estimating the critical shrinking factor of the singlet state subjected to local planar projective measurements and looking for the optimal set of measurements on each scenario. We use the search algorithm[2] presented on section 3.3.1. On each run of the search algorithm, we optimize over $N \in \{2, ..., 15\}$ planar projective measurements. The results are plotted as the blue points in fig. 4.1.

For all runs of the search algorithm with different random initial points the result for both the objective function η and optimization variables (measurements) were the same for all N. In all cases the optimal set of measurements found by the algorithm is the one in which the Bloch vectors of each measurement element are equally spaced on the plane, *i.e.*, each Bloch vector is separated from its next neighbors by an angle of $\frac{\pi}{N}$.

Next, we calculate lower bounds for the critical shrinking factor of the singlet state subjected to planar projective measurements using the method of outer polytope approximation described in section 3.3.3. Lower bounds were calculated for the cases of $N \in \{2, \ldots, 5\}$. The results are plotted as the red points in fig. 4.1. The lower bound for η found by the external polytope approximation matches up to 3–4 decimal places for all cases tested. This is enough evidence to guarantee that for the case of $N \in \{2, \ldots, 5\}$ planar projective measurements, the optimal set is the set of equally spaced measurements. We also conjecture this result to be valid for all values of $N \geq 2 \in \mathbb{N}$.

[2]The seesaw algorithm from section 3.3.2 is not suitable in this case since it is not possible to impose constraints to restrict the optimization over planar measurements or over planar projective measurements.

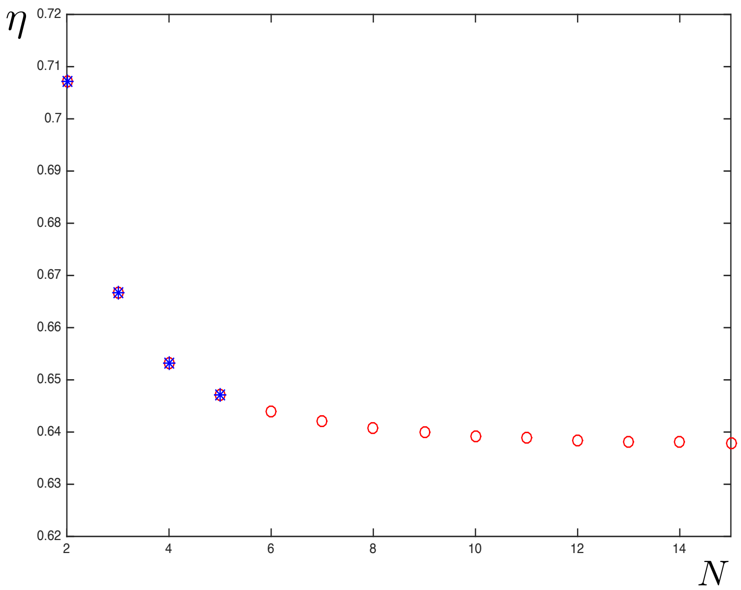

Figure 4.1: Plot of the upper bounds of the critical shrinking factor of the singlet state for N planar projective measurements calculated using the search algorithm from section 3.3.1 for $N \in \{2, \dots, 15\}$ (red circles) and plot of the lower bounds of the same problem calculated using the outer polytope approximation from section 3.3.3 for $N \in \{2, \dots, 5\}$ (blue stars).

This result motives us to study the family of equally spaced planar projective measurements with more detail. We aim to find the analytical value of the critical η as a function of the number of equally spaced planar projective measurements, namely, an expression of the form

$$\eta^* = f(N). \tag{4.8}$$

In order to do so, we will show that for $\eta \leq f(N)$, N equally spaced planar noisy projective measurements are jointly measurable and thus cannot exhibit steering. Then, we will show that for $\eta > f(N)$, N equally spaced planar projective measurements acting locally on the noisy singlet state violate a class of steering inequalities.

Let $\{E_{a|x}\}$, $x \in \{0, \dots, N-1\}$, $a \in \{-1, +1\}$ be noisy qubit projective measurements such that

$$E_{a|x} = \frac{1}{2} \left(\mathbb{1} + \eta \, a \, \hat{n}_x \cdot \vec{\sigma} \right) \tag{4.9}$$

where

$$\hat{n}_x = \left(\sin\left(\frac{\pi x}{N}\right), 0, \cos\left(\frac{\pi x}{N}\right)\right). \tag{4.10}$$

are equally spaced, coplanar, unit Bloch vectors.

By running the white noise robustness SDP (problem (3.33)) with assemblages generated by $\{E_{a|x}\}$ acting locally on the singlet state, for low values of N, the LHS models outputted by the SDPs provide an intuition on how to construct an ansatz to a mother-POVM. Our ansatz is the following 2^N–outcome POVM $\{G_{\vec{a}_\lambda}\}$, $\lambda \in \{1, \dots, 2^N\}$:

$$G_{\vec{a}_\lambda} = \begin{cases} \frac{1}{2N}(\mathbb{1} + \hat{v}_{\vec{a}_\lambda} \cdot \vec{\sigma}), & \text{for } \lambda \in \{1, \dots, 2N\} \\ 0, & \text{for } \lambda \in \{2N+1, \dots, 2^N\} \end{cases} \tag{4.11}$$

where $\vec{a}_\lambda = (a_{j=0}, ..., a_{j=N-1})$ is the outcome of $G_{\vec{a}_\lambda}$ and $a_j \in \{-1, +1\}$. The unit vector is given by $\hat{v}_{\vec{a}_\lambda} = (\sin\theta_\lambda, 0, \cos\theta_\lambda)$ with $\lambda \in \{1, \dots, 2N\}$ and the angle θ_λ is given by

$$\theta_k = \begin{cases} \frac{(\lambda - \frac{1}{2})\pi}{N}, & \text{for even } N \\ \frac{(\lambda-1)\pi}{N}, & \text{for odd } N. \end{cases} \tag{4.12}$$

The Bloch vectors of the ansatz of the mother-POVM are represented in green on fig. 4.2 and the Bloch vectors of the measurements $\{E_{a|x}\}$ are represented in pink. The Bloch vectors of $\{E_{a|x}\}$ are equally spaced on the plane and we always choose the vectors of the first measurement to be aligned with the z axis. The Bloch vectors of the ansatz to the mother-POVM are, in the case of even N, in the bisection of two neighbors $\{E_{a|x}\}$ vectors and, in the case of odd N, aligned with the $\{E_{a|x}\}$ vectors.

Now it is necessary to prove that: $\{G_{\vec{a}_\lambda}\}$ is indeed a POVM, showing that (i) its elements are all positive semidefinite, (ii) its elements sum to identity, and (iii) $\{G_{\vec{a}_\lambda}\}$ is a mother-POVM of $\{E_{a|x}\}$ for a certain value of η.

(i) $G_{\vec{a}_\lambda} \geq 0$, since its eigenvalues are 0 and $\frac{1}{N}$, $\forall\, \vec{a}_\lambda$.

(ii) $\displaystyle\sum_{\lambda=1}^{2^N} G_{\vec{a}_\lambda} = \sum_{\lambda=1}^{2N} \frac{1}{2N}\left(\mathbb{1} + \hat{v}_{\vec{a}_\lambda} \cdot \vec{\sigma}\right) + 0 = \frac{1}{2N}\left(2N\mathbb{1} + \sum_{\lambda=1}^{2N} \hat{v}_{\vec{a}_\lambda} \cdot \vec{\sigma}\right) = \mathbb{1}$, since $\displaystyle\sum_{\lambda=1}^{2N} \hat{v}_{\vec{a}_\lambda} = 0$.

(iii) $\displaystyle\sum_{\vec{a}_\lambda | a_x = a} G_{\vec{a}_\lambda} = \frac{1}{2N} \sum_{\vec{a}_\lambda | a_x = a} \left(\mathbb{1} + \hat{v}_{\vec{a}_\lambda} \cdot \vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + \frac{1}{N} \sum_{\vec{a}_\lambda | a_x = a} \hat{v}_{\vec{a}_\lambda} \cdot \vec{\sigma}\right)$.

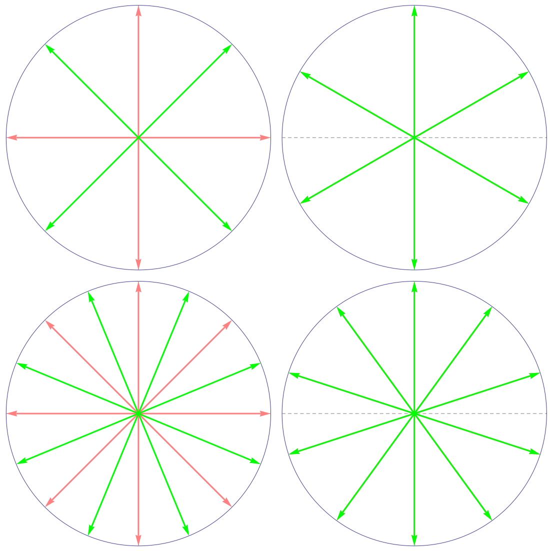

Figure 4.2: The direction of the N equally spaced Bloch vectors of the noisy projective measurements $\{E_{a|x}\}$ (plotted in pink) and the direction of the Bloch vectors of the ansatz of the mother-POVM $\{G_{\vec{a}_\lambda}\}$ (plotted in green). Notice that for odd N both pink and green vectors coincide.

In order to show that $\sum\limits_{\vec{a}_\lambda|a_x=a} G_{\vec{a}_\lambda} = E_{a|x}$ it is necessary that

$$\frac{1}{N} \sum_{\vec{a}_\lambda|a_x=a} \hat{v}_{\vec{a}_\lambda} = \eta\, a\, \hat{n}_x, \tag{4.13}$$

which is only true for some value of η.

Let us analise the left-hand side of eq. (4.13). The N vectors $\hat{v}_{(a_{j=1},\ldots,a_{j=N})}$ where $a_{j=x} = a$, are the vectors that lay on one of the halves of the plane determined by the axis perpendicular to $\hat{n}_x$, which half being determined by a. By summing over these vectors, the resulting vector is in the direction of $a\,\hat{n}_x$, since the components of $\hat{v}_{\vec{a}_k}$ that are perpendicular to $a\,\hat{n}_x$ cancel each other out. The norm of this vector is determined by

$$\left\| \sum_{\vec{a}_\lambda|a_x=a} \hat{v}_{\vec{a}_\lambda} \right\| = \sqrt{\left(\sum_{\vec{a}_\lambda|a_x=a} \sin\theta_\lambda \right)^2 + \left(\sum_{\vec{a}_\lambda|a_x=a} \cos\theta_\lambda \right)^2}. \tag{4.14}$$

Let us analise the even N and odd N case separately. For even N we have

$$\sum_{\vec{a}_\lambda|a_x=a} \sin\theta_\lambda = \sum_{\lambda=1}^{N} \sin\frac{(\lambda-1/2)\pi}{N} = \frac{1}{\sin\frac{\pi}{2N}}, \qquad \sum_{\vec{a}_\lambda|a_x=a} \cos\theta_\lambda = \sum_{\lambda=1}^{N} \cos\frac{(\lambda-1/2)\pi}{N} = 0, \tag{4.15}$$

resulting in

$$\left\| \sum_{\vec{a}_\lambda|a_x=a} \hat{v}_{\vec{a}_\lambda} \right\| = \sqrt{\frac{1}{\left(\sin\frac{\pi}{2N}\right)^2} + 0} = \frac{1}{|\sin\frac{\pi}{2N}|} = \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.16}$$

For odd N, we have

$$\sum_{\vec{a}_\lambda|a_x=a} \sin\theta_\lambda = \sum_{\lambda=1}^{N} \sin\frac{(k-1)\pi}{N} = \frac{1}{\tan\frac{\pi}{2N}}, \qquad \sum_{\vec{a}_\lambda|a_x=a} \cos\theta_\lambda = \sum_{\lambda=1}^{N} \cos\frac{(k-1)\pi}{N} = 1, \tag{4.17}$$

resulting in

$$\left\| \sum_{\vec{a}_\lambda|a_x=a} \hat{v}_{\vec{a}_\lambda} \right\| = \sqrt{\frac{1}{\left(\tan\frac{\pi}{2N}\right)^2} + 1} = \frac{1}{|\sin\frac{\pi}{2N}|} = \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.18}$$

On both cases we have obtained the same expression for the norm of the resulting vector. Hence,

$$\frac{1}{N} \sum_{\vec{a}_\lambda|a_x=a} \hat{v}_{\vec{a}_\lambda} = \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}\, a\, \hat{n}_x. \tag{4.19}$$

Comparing eq. (4.13) and eq. (4.19), we arrive at

$$\eta = \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.20}$$

This means that, for this value of η, $\{G_{\vec{a}_\lambda}\}$ will be a mother-POVM of $\{E_{a|x}\}$.

If for η in eq. (4.20) $\{E_{a|x}\}$ is jointly measurable, this set of measurements will also be joint measurable for any value of η smaller than that. Therefore, we have proven that the singlet state with shrinking factor satisfying

$$\eta \leq \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)} \tag{4.21}$$

is unsteerable for N equally spaced planar projective measurements. Now it is necessary to prove that the singlet state is steerable for shrinking factors that violate this inequality.

By running the dual problem of the white noise robustness with assemblages generated by $\{E_{a|x}\}$ acting locally on the singlet state, for low values of N, the steering inequalities outputted by the SDPs provide an intuition on how to construct a family of inequalities that will be violated whenever η violates ineq. (4.21). We are looking for inequalities in the form of

$$\beta = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}) \leq \beta^{\mathrm{uns}}. \tag{4.22}$$

The coefficients $\{F_{a|x}\}$ we propose are

$$F_{a|x} = \frac{1}{N} a\, \hat{n}_x \cdot \vec{\sigma}, \tag{4.23}$$

with $x \in \{0, \dots, N-1\}$ and $a \in \{-1, +1\}$, where the vectors $\hat{n}_x$, given by eq. (4.10), are unit vectors in the direction of the Bloch vectors of measurements $\{E_{a|x}\}$

We are looking for a bound for η above which these inequalities are violated. So let us calculate the value of the expression of β for a noisy assemblage:

$$\beta^{\eta} = \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma^{\eta}_{a|x}) \leq \beta^{\mathrm{uns}}, \tag{4.24}$$

where

$$\sigma^{\eta}_{a|x} = \eta \sigma_{a|x} + (1-\eta)\mathrm{Tr}(\sigma_{a|x})\frac{\mathbb{1}}{2}. \tag{4.25}$$

Substituting in eq. (4.24),

$$\beta^{\eta} = \eta \sum_{a,x} \mathrm{Tr}(F_{a|x}\, \sigma_{a|x}) + (1-\eta)\frac{1}{2} \sum_{a,x} \mathrm{Tr}(F_{a|x})\mathrm{Tr}(\sigma^{\eta}_{a|x}) \leq \beta^{\mathrm{uns}}. \tag{4.26}$$

Since all $F_{a|x}$ are traceless operators, the second term in the sum vanishes. To calculate the first term in the sum we need to know the form of the assemblages that are generated by equally spaced qubit projective measurements acting locally on the singlet state.

The singlet state is a pure maximally entangled two-qubit state. By performing local operations on Bob's qubit, the singlet state can be turned into any other pure maximally entangled state, such as $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. As proved on lemma 2, when Alice steers Bob, local operations performed by Bob on his subsystems do not alter the steerability of their state. Hence, we can calculate Bob's assemblage by either considering they hold the $|\Phi^+\rangle$ state or the $|\Psi^-\rangle$ (singlet) state. The assemblages will not be equal but their steerability will be the same, thus they are equivalent for our purpose. The advantage in considering the $|\Phi^+\rangle$ state is that the assemblages resulting from this state are given by eq. (2.20e),

$$\sigma_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1}\, |\Phi^+\rangle\langle\Phi^+|) = \frac{1}{2} M^T_{a|x}. \tag{4.27}$$

Hence the assemblage generated by $\{E_{a|x}\}$ on a pure maximally entangled two-qubit state can be written as

$$\sigma_{a|x} = \mathrm{Tr}_A(E_{a|x} \otimes \mathbb{1}|\Psi^-\rangle\langle\Psi^-|) = \frac{1}{4}(\mathbb{1} + a\,\hat{n}_x \cdot \vec{\sigma}). \tag{4.28}$$

Calculating the first term in eq. (4.24), we have

$$\begin{aligned}
\sum_{a,x} \mathrm{Tr}(F_{a|x}\,\sigma_{a|x}) &= \sum_{a,x} \frac{1}{4N} \mathrm{Tr}\left[(a\,\hat{n}_x \cdot \vec{\sigma})\,(\mathbb{1} + a\,\hat{n}_x \cdot \vec{\sigma})\right] && \text{(4.29a)}\\
&= \sum_{a,x} \frac{1}{4N} \mathrm{Tr}\left[(a\,\hat{n}_x \cdot \vec{\sigma}) + (a\,\hat{n}_x \cdot \vec{\sigma})^2\right] && \text{(4.29b)}\\
&= \frac{1}{4N} \sum_{a,x} \mathrm{Tr}(a\,\hat{n}_x \cdot \vec{\sigma}) + \frac{1}{4N} \sum_{a,x} \mathrm{Tr}(\mathbb{1}) && \text{(4.29c)}\\
&= \frac{1}{4N} 0 + \frac{1}{4N} 4N = 1, && \text{(4.29d)}
\end{aligned}$$

which implies

$$\beta^{\eta} = \eta \leq \beta^{\mathrm{uns}}. \tag{4.30}$$

The last step is to calculate the unsteerable bound β^{uns} as a function of the number of measurements N. This is possible with some help from a constraint of the dual SDPs that provide steering inequalities. The constraint that interests us is of the form

$$\sum_{a,x} D_\lambda(a|x) F_{a|x} \leq \beta^{\mathrm{uns}}\mathbb{1} \quad \forall\, \lambda. \tag{4.31}$$

To saturate this inequality, the operators $U_\lambda = \sum_{a,x} D_\lambda(a|x) F_{a|x}$ must have their highest eigenvalue equal to β^{uns}, so let us calculate the eigenvalues of $\{U_\lambda\}$.

$$\begin{aligned}
U_\lambda &= \frac{1}{N} \sum_{a,x} D_\lambda(a|x) a\,\hat{n}_x \cdot \vec{\sigma} && \text{(4.32a)}\\
&= \frac{1}{N} \sum_{a,x} \vec{u}_\lambda \cdot \vec{\sigma}, && \text{(4.32b)}
\end{aligned}$$

where $\vec{u}_\lambda = D_\lambda(a|x) a\,\hat{n}_x$ and the eingenvalues of U_λ are $\pm\frac{\|\vec{u}_\lambda\|}{N}$. For different λ, $\vec{u}_\lambda$ will have different norms. The vector with largest norm, which will yield largest eigenvalue among U_λ, is the vector that corresponds to summing N vectors $a\hat{n}_x$ that are neighbors on the plane, for example, all vectors $x \in \{0, \ldots, N-1\}$ with $a = +1$. So, it is true that

$$\max_{\lambda} \|\vec{u}_{\lambda}\| = \left\| \sum_{x=0}^{N-1} +\hat{n}_x \right\| \tag{4.33a}$$

$$= \sqrt{\left(\sum_{x=0}^{N-1} \sin \frac{\pi x}{N} \right)^2 + \left(\sum_{x=0}^{N-1} \cos \frac{\pi x}{N} \right)^2}. \tag{4.33b}$$

The argument of the sine and cosine functions in the equation above is exactly θ_λ from eq. (4.12) for odd N. Hence, eq. (4.33b) equals eq. (4.14) and

$$\max_{\lambda} \|\vec{u}_{\lambda}\| = \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.34}$$

The maximum eigenvalue of U_λ among all λ, which is the unsteerable bound of the ineq. (4.24), is

$$\beta^{\text{uns}} = \frac{1}{N} \max_{\lambda} \|\vec{u}_{\lambda}\| = \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.35}$$

Rewriting ineq. (4.30) we arrive at the steering inequality

$$\beta^{\eta} = \sum_{a,x} \text{Tr}(F_{a|x}\, \sigma^{\eta}_{a|x}) \leq \beta^{\text{uns}} \implies \eta \leq \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.36}$$

Thus, we have proven that for $\eta > \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}$ the singlet state is steerable by showing it violates the above inequality when equally spaced planar projective measurements are locally performed on it.

Finally, we have arrived at the function $f(N)$ from eq. (4.8):

$$\eta^* = \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)}. \tag{4.37}$$

This is the critical shrinking factor of the singlet state when subjected to N equally spaced planar projective measurements. By taking the limit where $N \to \infty$ it is possible to calculate a bound below which any set of arbitrarily many planar projective measurements are jointly measureable:

$$\eta_{\infty} = \lim_{N \to \infty} \frac{1}{N \sin\left(\frac{\pi}{2N}\right)} = \frac{2}{\pi}. \tag{4.38}$$

The function $f(N)$ matches the values found by the numerical optimization in the case of 2 to 15 measurements. On fig. 4.3 the result of the numerical optimization is plotted (same as in fig. 4.1) in red circles, along with the value of $f(N)$ in blue 'x'. The dotted line is the limit $\eta_{\infty} = \frac{2}{\pi}$.

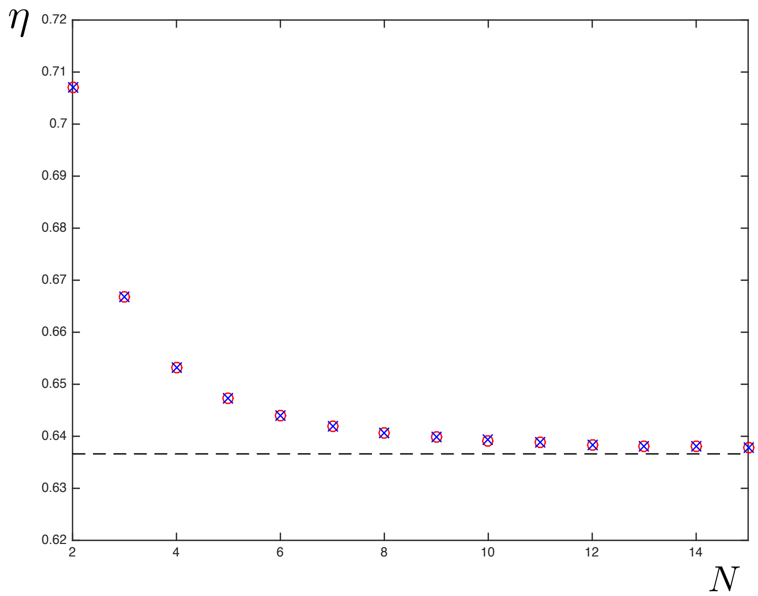

Figure 4.3: Plot of the result of numerical optimizations for the critical shrinking factor of $N \in \{2, \ldots, 15\}$ planar projective measurements acting locally on the singlet state (red circles). Value of the function $\frac{1}{N \sin\left(\frac{\pi}{2N}\right)}$ for $N \in \{2, \ldots, 15\}$ (blue 'x's). Limit $\eta_\infty = \frac{2}{\pi}$ below which all planar noisy qubit projective measurements are jointly measurable (black dotted line). The data coincides for all value of N plotted.

We can enunciate these results from the point of view of joint measurability and, equivalently, from the point of view of steering.

Result 1. *The qubit Werner states are unsteerable with sets of N equally spaced planar projective measurements if, and only if*

$$\eta \leq \frac{1}{N} \frac{1}{\sin\left(\frac{\pi}{2N}\right)} \tag{4.39}$$

and unsteerable for any set of arbitrarily many planar projective measurements if, and only if,

$$\eta \leq \frac{2}{\pi}. \tag{4.40}$$

Result 2. *The sets of N equally spaced planar noisy qubit projective measurements are jointly measurable if, and only if, η satisfies eq.* (4.39)*. Any set of arbitrarily many noisy qubit projective measurements is jointly measurable if η satisfies eq.* (4.40)*.*

This result is equivalent to the one presented on reference [108], where experiments involving equally spaced projective qubit measurements have been reported. It has also been independently derived by the authors of reference [109].

The demonstration presented here does not prove that any set of N (not necessarily equally spaced) planar measurements is jointly measurable below the threshold defined in eq. (4.37). This is still an open problem. However, from the results of our numerical calculations we make the following conjecture, written in two equivalent forms:

Conjecture 1. *The qubit Werner states with η satisfying eq.* (4.39) *are unsteerable for any set of N planar projective measurements.*

Conjecture 2. *All sets of N planar noisy qubit projective measurements are jointly measurable if η satisfies eq.* (4.39).

One possible way of proving these conjectures would be to show that there exist a mother-POVM or LHS model for all other possible sets of N planar projective measurements.

4.2 General projective measurements

After exploring and learning from the case of planar qubit measurements we move on to the case of general qubit projective measurements, the ones whose Bloch vectors do not have any restriction on direction. We begin by investigating the case of projective measurements before general POVMs since the numerical optimization in this case is less costly.

Calculations were ran for the case of $N \in \{2, \dots, 13\}$ projective measurements acting locally on the singlet state using the search method (section 3.3.1) for finding upper bounds on the critical shrinking factor. For the case of $N \in \{2, \dots, 5\}$ projective measurements, lower bounds were calculated using the method of outer polytope approximation (section 3.3.3). The results are plotted on fig. 4.4.

Since the optimal sets of measurements in the case of planar measurements were the sets in which all Bloch vectors were equally spaced from their next neighbors, we hypothesize that some notion of equally spaced points on a sphere could correspond to

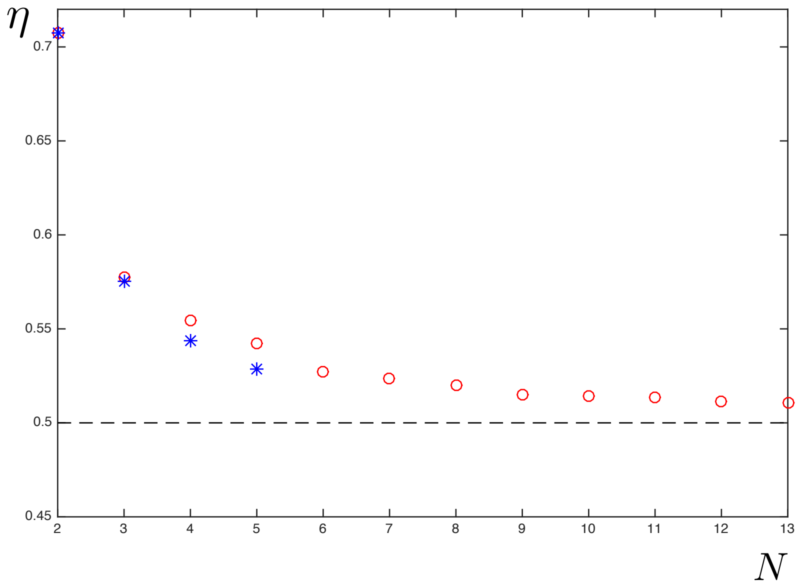

Figure 4.4: Plot of the upper bounds of the critical shrinking factor of the singlet state for N projective measurements calculated using the search algorithm from section 3.3.1 for $N \in \{2, \ldots, 13\}$ (red circles). Plot of the lower bounds of the same problem calculated using the outer polytope approximation from section 3.3.3 for $N \in \{2, \ldots, 5\}$ (blue stars). Plot of the bound below which the singlet state is unsteerable for any set of arbitrarily many projective measurements (black dotted line).

the direction of the Bloch vectors of the optimal sets of general projective measurements. We ran the white noise robustness SDP for the case of fixed local measurements on the singlet state for two families of measurements that correspond to two different notions of equally spaced points on a sphere[3]: the Thomson distribution [110] and the Fibonacci distribution [111]. The SDP was ran for up to 18 measurements, a higher value than for the search algorithm, since in this case only one SDP is needed and not an optimization over SDP results. The results are plotted on fig. 4.5 and resumed on table 4.1.

Immediately, it is possible to see that neither candidates for the optimal sets are the actual solution. In all cases, the heuristic search algorithm was able to find better candidates for the optimal sets of measurements, *i.e.*, sets of measurements that exhibit higher steerability. Most distributions of equally spaced vectors on a sphere are only

[3]More precisely, in this case it is necessary to equally distribute N points on a half-sphere and take the points in the antipodal direction to construct N projective measurements.

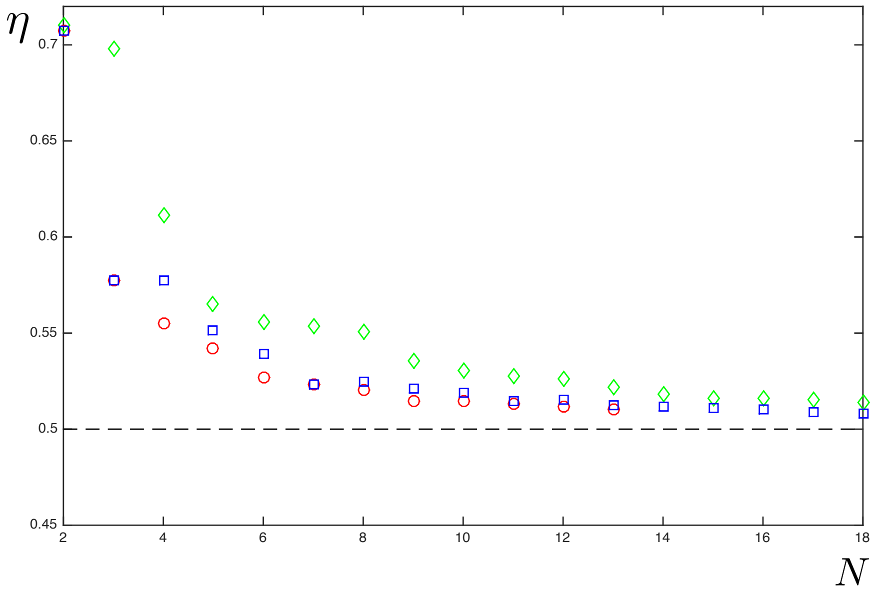

Figure 4.5: Plot of the upper bounds of the critical shrinking factor of the singlet state for N projective measurements calculated using the search algorithm from section 3.3.1 for $N \in \{2, \ldots, 13\}$ (red circles). Plot of the shrinking factor of the set of measurements constructed from the Fibonacci distribution of equally spaced points on a half-sphere for $N \in \{2, \ldots, 18\}$ measurements (green diamonds). Plot of the shrinking factor of the set of measurements constructed from the Thomson distribution of equally spaced points on a half-sphere for $N \in \{2, \ldots, 18\}$ measurements (blue squares). Plot of the bound below which the singlet state is unsteerable for any set of arbitrarily many projective measurements (black dotted line).

effective in the case of large number of points [112]. For low N, we have not found other distributions that could generate good candidates for the optimal sets.

Still, it is possible to examine the sets of measurements that yield the upper bound given by the search algorithm and look for a pattern. These measurement sets are plotted as Bloch vectors on fig. 4.6. Even though for up to 6 measurements there appears to be a pattern in the distribution of the Bloch vectors, given by 3 or less coplanar equally spaced vectors plus an agglomeration of the remaining vectors around the poles of the Bloch sphere, this pattern, unfortunately, does not hold for 7 or more measurements.

Overall, our results state that the critical shrinking factor of the singlet state for projective measurements lies in the interval $\eta^* \in (0.5000, 0.5079]$, the last value being achieved by performing a set of 18 measurements. A summary of our numerical calcula-

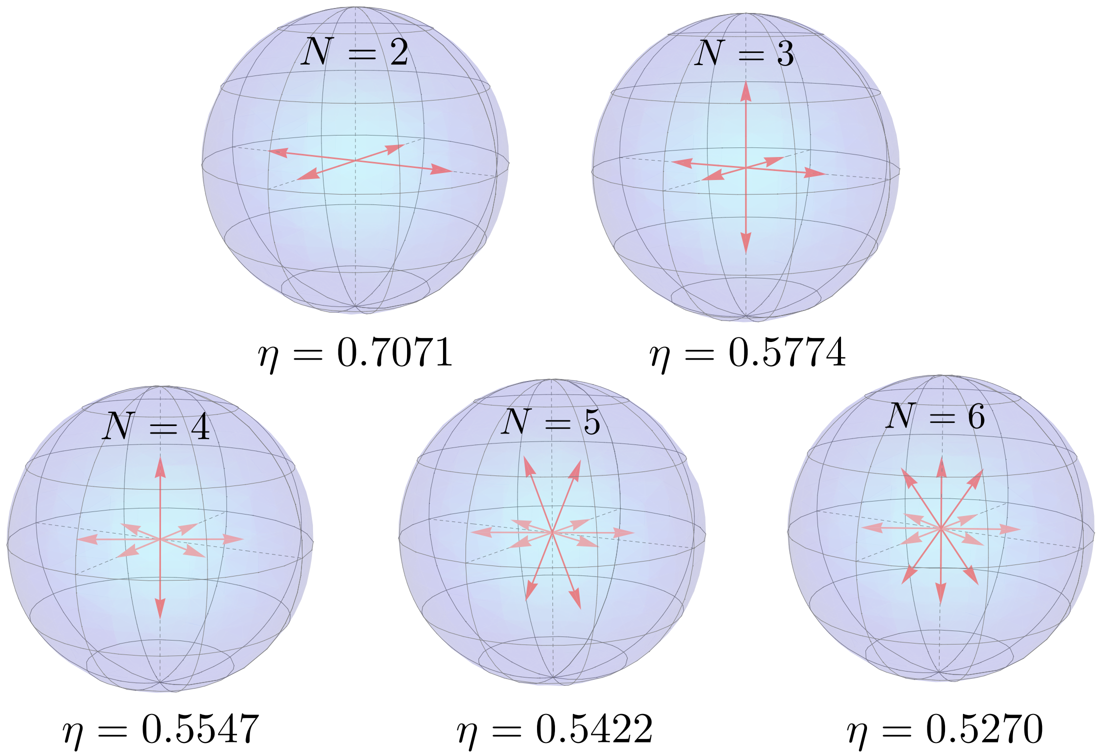

Figure 4.6: Bloch vectors of the solution of the search algorithm (section 3.3.1) of possible optimal sets of N projective measurements.

tions is presented in table 4.1. In the next section, we explore the use of general POVMs to exhibit steering on quantum states.

4.3 General POVM relevance

There are known cases of general POVMs being relevant in nonlocality scenarios. For example, in reference [113] there has been reported a Bell inequality that, for fixed dimension or state, can only be maximally violated by the use of non-projective measurements. For the case of steering, there has not been reported, to the extent of our knowledge, an analogous inequality, in which general POVMs are relevant over projective measurements.

It could be possible that the rich structure of general POVMs would allow the

Projective qubit measurements

	Sphere Opt.		Planar Opt.		Fixed sets	
N	Upper	Lower	Upper	Lower	Thomson	Fibonnaci
2	0.7071	0.7071	0.7071	0.7071	0.7071	0.7102
3	0.5774	0.5755	0.6667	0.6667	0.5774	0.6981
4	0.5547	0.5437	0.6533	0.6532	0.5774	0.6114
5	0.5422	0.5283	0.6472	0.6470	0.5513	0.5653
6	0.5270		0.6440		0.5393	0.5561
7	0.5234		0.6420		0.5234	0.5533
8	0.5202		0.6407		0.5250	0.5508
9	0.5149		0.6399		0.5209	0.5359
10	0.5144		0.6392		0.5191	0.5302
11	0.5132		0.6388		0.5148	0.5274
12	0.5117		0.6384		0.5152	0.5261
13	0.5105		0.6382		0.5126	0.5220
14			0.6380		0.5114	0.5180
15			0.6378		0.5107	0.5158
16					0.5106	0.5158
17					0.5086	0.5150
18					0.5079	0.5136

Table 4.1: Summary of the results of our numerical optimizations for projective measurements.

demonstration of steering and nonlocality in such a way that is not possible for projective measurements. This problem of general POVM relevance is discussed by Werner in his 2015 article [114], without arriving at an ultimate conclusion. In his seminal paper [27] from 1989, Werner constructed an unsteering (LHS) model (though it was not called by these terms yet) for the qubit Werner state $\eta = \frac{1}{2}$ subjected to any arbitrary number of projective measurements, proving that Werner states are *unsteerable for all projective measurements* with $\eta \leq \frac{1}{2}$. The work of Barrett in reference [46] provides a locality (LHV) model for the Werner state with $\eta = \frac{5}{12}$ for any arbitrary number of POVMs, proving that Werner states with $\eta \leq \frac{5}{12}$ are *local for all general POVMs*. Subsequently, the authors of reference [44] proved Barrett's locality model to be also an unsteering model, implying that Werner states with $\eta \leq \frac{5}{12}$ are *unsteerable for all general POVMs*.

In his 2015 paper [114], Werner mentions that at the time his unsteering model for projective measurements was published, he already had the concern of extending it to general POVMs but has not succeeded in doing so up to this date. He mentions no one has been able to show whether or not Barrett's model provides the best bound to the unsteerability of Werner states subjected to an infinite amount of general POVMs nor show if his model could be extended for general POVMs, which if proved would imply that all states that are unsteerable for all projective measurements are also unsteerable for all POVMs. These important unachieved results would path the way to determining the relevance of the use of general measurements for quantum correlations.

In the work of this dissertation, we have performed an extensive investigation of the behavior of quantum states in steering scenarios when subjected to general POVMs. In order to do so, we used the seesaw algorithm from section 3.3.2 to calculated upper bounds on the steerability of the qubit Werner states, by optimizing over all possible sets of $N \in \{2, \ldots, 13\}$ POVMS. We have ran this optimization for general 4-outcome POVMs, since as mentioned in section 3.3.1, there only exist extremal qubit POVMs with up to 4 outcomes.

After many runs of the seesaw algorithm, in **no occasion** have we found a set of general POVMs that is more robust than the set of projective measurements found by the search algorithm. For all investigated values of N we have been able to reach the bound provided by projective measurements (fig. 4.4 and table 4.1) but never surpass it. This is strong evidence that for steering, general POVMs are not more useful than projective measurements.

Even though we have not arrived at a definite conclusion, we consider the investigation of the relevance of general POVMs for steering to be our most important result. From the evidence we obtained, we make the following conjecture:

Conjecture 3. *The qubit Werner states are unsteerable for all sets of arbitrarily many general POVMs if, and only if,* $\eta \leq \frac{1}{2}$.

To prove this conjecture would be to prove that POVMs are not more relevant to steering than projective measurements. It would also prove that Werner's model is extendable for general POVMs.

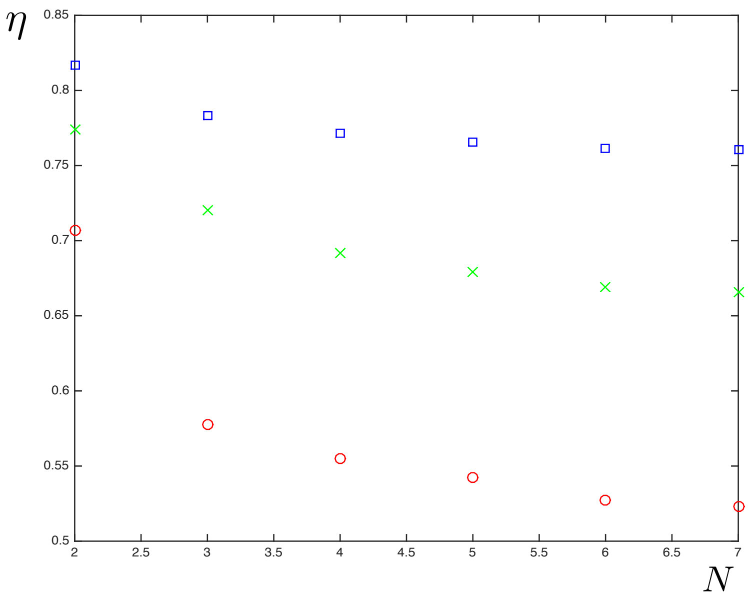

Figure 4.7: Upper bound of the critical shrinking factor of $N \in \{2, \ldots, 7\}$ projective (red circles), regular trine (green 'x's) POVM, and regular tetrahedron (blue squares) SIC-POVM, performed locally on the singlet state, as found by the search algorithm.

To deepen our analysis, we have ran calculations for the case of two particularly interesting POVMs: the regular trine POVM, a 3-outcome POVM whose elements have the same trace and whose Bloch vectors are coplanar, equally spaced, and have the same norm; and the regular tetrahedron POVM, a 4-outcome POVM whose elements also have the same trace and whose Bloch vectors are in the direction of the vertices of a regular tetrahedron and have the same norm. Both of these POVMS are extremal measurements and are interesting POVMs because they are the planar qubit symmetric informationally complete (SIC)-POVM and the qubit SIC-POVM, respectively [115]. The statistics resulting from a SIC-POVM are able to determine completely the quantum state in which the measurement was performed. They are measurements with high experimental interest and find many applications in quantum state tomography [116].

We have ran calculations using the search algorithm to estimate the critical η of the qubit Werner states when subjected to a set of N regular tetrahedron POVMs or N regular trine POVMs for $N \in \{2, \ldots, 7\}$. The optimization becomes much harder for a larger number of measurements, specially for POVMs with more than 2 outcomes.

Qubit projective measurements and SIC-POVMs

N	Projective	Trine	Tetrahedron
2	0.7071	0.7739	0.8165
3	0.5774	0.7202	0.7829
4	0.5547	0.6917	0.7716
5	0.5422	0.6791	0.7653
6	0.5270	0.6690	0.7617
7	0.5234	0.6656	0.7605

Table 4.2: Summary of the results of our numerical optimizations for projective measurements and SIC-POVMs.

The results are plotted in fig. 4.7 and listed on table 4.2. For the case of 2 regular trines and 2 regular tetrahedrons measurements, the Bloch vectors of the optimal sets of measurements found by the search algorithm are plotted in fig. 4.8. The Bloch vectors of the optimal set in both cases corresponds to antipodal vectors.

Clearly, compared to the result from the previous section, the SIC-POVMs are far less effective in exhibiting steering than projective measurements, which is further evidence to support our conjecture 3.

Along the way, we have encountered evidences of some other possible properties of projective and non-projective qubit measurements that are worth mentioning:

(i) Adding a projective measurement in any direction to a set of N projective measurements always increases the robustness of the set, even if the extra projective measurement is arbitrarily close to any other measurement in the initial set.

(ii) Adding a regular tetrahedron POVM to the set of optimal projective measurements found by the search algorithm for $N \leq 6$ (fig. 4.6), in any direction, never increases the robustness of the set.

(iii) Adding a regular tetrahedron POVM to a set of non-optimal measurements always increases the robustness of the set.

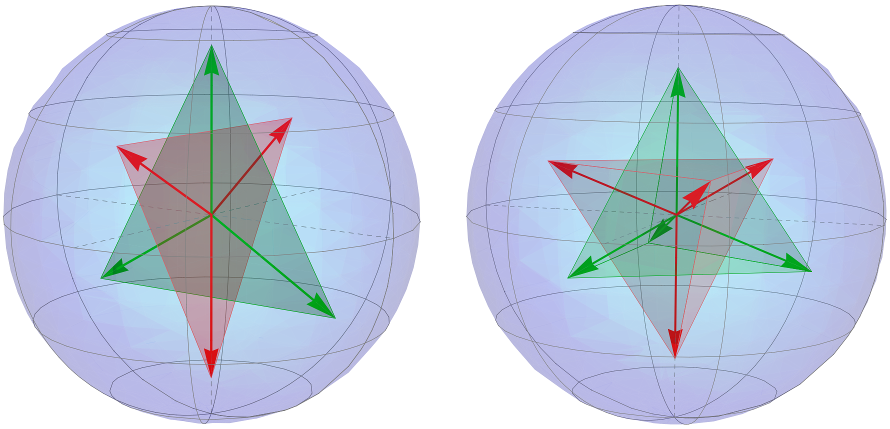

Figure 4.8: Bloch vectors of the optimal set of 2 regular trines (left) POVM and 2 regular tetrahedrons (right) SIC-POVM found by the search algorithm (section 3.3.1).

(iv) The critical shrinking factor of a quantum state as a function of the number of measurements being performed locally on it appears to be a strictly monotonically decreasing function.

(v) For higher dimensional states, the robustness is increased by performing projective measurements with more than 2 outcomes and maximal robustness appears to be exhibitted by projective measurements with $O_A = d$. General POVMs with $O_A > d$ seem to be not relevant.

4.4 Further results

We have presented our results on steering with a finite number of measurements applying our methods to the singlet state and calculating its white noise robustness.

It is important to highlight that, even though this is a particular kind of quantum state since it is pure, maximally entangled, and symmetric, these methods are applicable to any other quantum state of any dimension. Of course, for high dimension the computational cost may be unviable. But for qubit states the complexity is the same as in the calculations reported here.

In the following, we list some preliminary results and new projects that will follow from the work of this dissertation:

- We have tested our methods for estimating the critical shrining factor of quantum states subjected to a finite number of measurements for higher-dimensional isotropic states, random states, partly entangled states, and one-way steerable states. Our methods have shown to be effective in all cases.

- The calculations presented here for the white noise robustness of steering have been redone for the generalized robustness of steering using the seesaw method, also for the singlet state. The most interesting result so far is that, for all cases studied (scenarios ranging from 2 to 13 POVMs), the optimal set of measurements for white noise robustness is the same set of measurements that is optimal for the generalized robustness of steering. This result does not extend, however, to other quantum states. A counter-example was found when studying partly entangled qubit states.

- Still using the generalized robustness of steering, we applied the seesaw algorithm to estimating the steerability of all possible bipartitions of the quantum state reported in reference [117]. On this article the authors claim to have simulated the evolution of a closed system composed of two spatially separated pairs of qubits, each pair constituted by an ancila and a system qubit. They report entanglement quantification. Using our methods, we were able to analyze the evolution of the steerability that could be demonstrated by suitable measurements on all possible bipartitions of the 4-qubit state simulated on the experiment of reference [117]. We have found steps in the evolution were the states are entangled but unsteerable, one-way steerable, and finally two-way steerable. According to our calculations, the dynamics of the steerability in this case goes as follows. First, steering is only possible between the two systems. Next, one-way steering becomes possible from each system to its ancila. Following, two-way steering is possible and the ancilas are also able to steer its systems. Then, one-way steering is possible again, but now from the ancilas to its systems. Finally, steering is only possible between the two ancilas. This behavior follows the redistribution of entanglement, initially in the reduced state of the two systems to eventually the reduced state of the two ancilas.

- An extension of this work would be to reformulate these methods to the study of Bell nonlocality. Deciding whether or not a set of probability distributions is nonlocal is a linear programming problem, a particular case of semidefinite programming problem. Hence, the methods presented in this dissertation can be adapted for this kind of problem as well. As discussed in chapter 1, it is known that the Werner state is nonlocal for two measurements being performed locally by each party for $\eta > \frac{1}{\sqrt{2}}$. It is also known that there is an inequality that is violated by 465 measurements on each side for a value of η slightly less than $\frac{1}{\sqrt{2}}$ [50]. In the regime of few measurements, it is not know whether or not nonlocality can be demonstrated for smaller values of η. New results in this direction would help better approximate the Grothendieck constant [48] (see section 1.7) and reveal other useful Bell inequalities aside from the CHSH inequality [39].

Final Remarks

This dissertation has presented an introduction to the topic of quantum steering. We have presented and discussed the basic concepts behind the definition of entanglement, steering, and nonlocality, always stressing the relationship between these correlations. The similarities between them helped us put on firmer ground the significance of local models and witnesses. The differences between them assisted us on establishing a hierarchy between these correlation and also on approaching the phenomenon of one-way steering. A deeper study of quantum steering allowed us to recognize its tight relationship with measurement incompatibility. We explored this connection to achieve most part of our results.

If you have made this far, we hope you have finished this text with a better understanding of quantum correlations and, specially, a more concrete idea of what is quantum steering. We also hope to have highlighted the most interesting aspects in the theory of steering, and that it was possible for you to benefit from our exposition somehow.

As for what concerns the tools used to study steering and to approach some open problems, semidefinite programming was showed to be the most valuable one. From a fundamental point of view, the formulation of problems in steering theory as SDPs helped us gain intuition about the problems at hand. This was true specially when calculating the white noise robustness of quantum states subjected to local planar projective measurements. On this problem specifically, the results of the SDP calculations brought insights on how to solve the problem analytically. For the more difficult problems of general projective measurements and general POVMs, where analytical solutions were out of reach, methods of numerical nonlinear optimization based on SDPs allowed us to find approximate solutions for the scenarios that involved a small number of measurements

– which are scenarios with high experimental interest. Our methods were proved to be practicable and efficient, and easily adaptable to different problems.

As far as our results go, we finish with many open questions. The optimal sets of measurements and the true relevance of general POVMs for steering are still unknown. Yet, our advances provide evidences that projective measurements are optimal for steering. We hope this work can support further research on this topic.

Bibliography

[1] E. Schrödinger. Discussion of probability relations between separated systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, **31**, 555–563, (1935).

[2] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, **47**, 777–780, (1935).

[3] H. M. Wiseman, S. J. Jones, and A. C. Doherty. Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox. *Phys. Rev. Lett.*, **98**, 2, (2007). [arXiv: quant-ph/0612147].

[4] M. A. Nielsen and I. L. Chuang. Quantum computation and quantum information. Cambridge University Press, (2010).

[5] John Watrous. Theory of quantum information and lecture notes. (Accessed on May 2016).

[6] K. Vogel and H. Risken. Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase. *Phys. Rev. A*, **40**, 2847–2849, (1989).

[7] Stefan Weigert. Pauli problem for a spin of arbitrary length: A simple method to determine its wave function. *Phys. Rev. A*, **45**, 7688–7696, (1992).

[8] Cyril Branciard, Eric G. Cavalcanti, Stephen P. Walborn, Valerio Scarani, and Howard M. Wiseman. One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering. *Phys. Rev. A*, **85**, 010301, (2012). [arXiv: 1109.1435].

[9] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Rev. Mod. Phys.*, **86**, 419–478, (2014).

[10] Daniel Cavalcanti and Paul Skrzypczyk. Quantum steering: a short review with focus on semidefinite programming. (2016). [arXiv: 1604.00501].

[11] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters. Quantum nonlocality without entanglement. *Phys. Rev. A*, **59**, 1070–1091, (1999). [arXiv: quant-ph/9804053].

[12] Max Born. Zur Quantenmechanik der Stossvorgänge. *(English translation: On the quantum mechanics of collisions, in Quantum theory and measurement, section I.2, J. A. Wheeler and W. H. Zurek, Princeton University Press, 1983). Zeitschrift für Physik*, **37**, 863–867, (1926).

[13] Scott Aaronson. Lecture notes. (Accessed on Jul 2016).

[14] E. Schrödinger. Probability relations between separated systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, **32**, 446–452, (1936).

[15] Nicolas Gisin. Stochastic quantum dynamics and relativity. *Helvetica Physica Acta*, **62**, 363, (1989).

[16] Ana Belén Sainz, Nicolas Brunner, Daniel Cavalcanti, Paul Skrzypczyk, and Tamás Vértesi. Postquantum steering. *Phys. Rev. Lett.*, **115**, 190403, (2015). [arXiv: 1505.01430].

[17] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, **81**, 865–942, (2009). [arXiv: quant-ph/0702225].

[18] Marcelo de Oliveira Terra Cunha. Emaranhamento: caracterização, manipulação e consequências. PhD thesis, Programa de Pós-Graduação em Física - UFMG, (2005).

[19] Asher Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, **77**, 1413–1415, (1996). [arXiv: quant-ph/9604005].

[20] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Separability of mixed states: necessary and sufficient conditions. *Phys. Lett. A*, **223**, 1–8, (1996). [arXiv: quant-ph/9605038].

[21] Leonid Gurvits. Special issue on STOC 2003 classical complexity and quantum entanglement. *Journal of Computer and System Sciences*, **69**, 448 - 484, (2004). [arXiv: quant-ph/0303055].

[22] Barbara M. Terhal. Bell inequalities and the separability criterion. *Phys. Lett. A*, **271**, 319–326, (2000). [arXiv: quant-ph/9911057].

[23] Gláucia Murta. Não-localidade em sistemas quânticos. Master's thesis, Programa de Pós-Graduação em Física - UFMG, (2012).

[24] Marco Túlio Quintino. Black box correlations: Locality, noncontextuality, and convex polytopes. Master's thesis, Programa de de Pós-Graduação em Física - UFMG, (2012).

[25] Matthew F. Pusey. Negativity and steering: A stronger Peres conjecture. *Phys. Rev. A*, **88**, 032313, (2013). [arXiv: 1305.1767].

[26] Paul Skrzypczyk, Miguel Navascués, and Daniel Cavalcanti. Quantifying Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **112**, 180404, (2014). [arXiv: 1311.4590].

[27] Reinhard F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, **40**, 4277–4281, (1989).

[28] M. Froissart. Constructive generalization of Bell's inequalities. *Il Nuovo Cimento B (1971-1996)*, **64**, 241–251, (1981).

[29] Itamar Pitowsky. Correlation polytopes: Their geometry and complexity. *Mathematical Programming*, **50**, 395–414, (1991).

[30] Gláucia Murta. Bounds on quantum nonlocality. PhD thesis, Programa de Pós-Graduação em Física - UFMG, (2016).

[31] Leonardo Guerini de Souza. Local models and hidden nonlocality in quantum theory. Master's thesis, Programa de de Pós-Graduação em Matemática - UFRGS, (2014). [arXiv: 1407.5958].

[32] Remigiusz Augusiak, Maciej Demianowicz, and Antonio Acín. Local hidden-variable models for entangled quantum states. *Journal of Physics A: Mathematical and Theoretical*, **47**, 424002, (2014). [arXiv: 1405.7321].

[33] John S. Bell. On the Eintein Podolsky Rosen paradox. *Physics 1*, 195–200, (1964).

[34] Bárbara Lopes Amaral. The exclusivity principle and the set of quantum correlations. PhD thesis, Programa de Pós-Graduação em Matemática - UFMG, (2014).

[35] Alain Aspect, Philippe Grangier, and Gérard Roger. Experimental realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A new violation of Bell's inequalities. *Phys. Rev. Lett.*, **49**, 91–94, (1982).

[36] B. Hensen, H. Bernien, A. E. Dreau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellan, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson. Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature*, **526**, 682–686, (2015). [arXiv: 1508.05949].

[37] Marissa Giustina, Marijn A. M. Versteegh, Sören Wengerowsky, Johannes Handsteiner, Armin Hochrainer, Kevin Phelan, Fabian Steinlechner, Johannes Kofler, Jan-Åke Larsson, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Morgan W. Mitchell, Jörn Beyer, Thomas Gerrits, Adriana E. Lita, Lynden K. Shalm, Sae Woo Nam, Thomas Scheidl, Rupert Ursin, Bernhard Wittmann, and Anton Zeilinger. Significant-loophole-free test of Bell's theorem with entangled photons. *Phys. Rev. Lett.*, **115**, 250401, (2015). [arXiv: 1511.03190].

[38] Lynden K. Shalm, Evan Meyer-Scott, Bradley G. Christensen, Peter Bierhorst, Michael A. Wayne, Martin J. Stevens, Thomas Gerrits, Scott Glancy, Deny R. Hamel, Michael S. Allman, Kevin J. Coakley, Shellee D. Dyer, Carson Hodge, Adriana E. Lita, Varun B. Verma, Camilla Lambrocco, Edward Tortorici, Alan L. Migdall, Yanbao Zhang, Daniel R. Kumor, William H. Farr, Francesco Marsili, Matthew D. Shaw, Jeffrey A. Stern, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Thomas Jennewein, Morgan W. Mitchell, Paul G. Kwiat, Joshua C. Bienfang, Richard P. Mirin, Emanuel Knill, and Sae Woo Nam. Strong loophole-free test of local realism. *Phys. Rev. Lett.*, **115**, 250402, (2015). [arXiv: 1511.03189].

[39] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, **23**, 880–884, (1969).

[40] Joseph Bowles, Tamás Vértesi, Marco Túlio Quintino, and Nicolas Brunner. One-way Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **112**, 200402, (2014). [arXiv: 1402.3607].

[41] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid. Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox. *Phys. Rev. A*, **80**, 032112, (2009). [arXiv: 0907.1109].

[42] Paul Skrzypczyk and Daniel Cavalcanti. Loss-tolerant Einstein-Podolsky-Rosen steering for arbitrary-dimensional states: Joint measurability and unbounded violations under losses. *Phys. Rev. A*, **92**, 022354, (2015). [arXiv: 1502.04926].

[43] Eric G. Cavalcanti, Christopher J. Foster, Maria Fuwa, and Howard M. Wiseman. Analog of the Clauser-Horne-Shimony-Holt inequality for steering. *J. Opt. Soc. Am. B*, **32**, A74-A81, (2015). [arXiv: 1412.8178].

[44] Marco Túlio Quintino, Tamás Vértesi, Daniel Cavalcanti, Remigiusz Augusiak, Maciej Demianowicz, Antonio Acín, and Nicolas Brunner. Inequivalence of entanglement, steering, and Bell nonlocality for general measurements. *Phys. Rev. A*, **92**, 032107, (2015). [arXiv: 1501.03332].

[45] Nicolas Gisin. Bell's inequality holds for all non-product states. *Physics Letters A*, **154**, 201–202, (1991).

[46] Jonathan Barrett. Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality. *Phys. Rev. A*, **65**, 042302, (2002). [arXiv: 0107045].

[47] Antonio Acín, Nicolas Gisin, and Benjamin Toner. Grothendieck's constant and local models for noisy entangled quantum states. *Phys. Rev. A*, **73**, 062105, (2006). [arXiv: 0606138].

[48] Alexander Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques. *Bol. Soc. Mat. Sao Paulo*, **8**, 1–79, (1953).

[49] Steven R. Finch. Mathematical constants, 235-237. Cambridge University Press, (2016).

[50] Tamás Vértesi. More efficient Bell inequalities for Werner states. *Phys. Rev. A*, **78**, 032112, (2008). [arXiv: 0806.0096].

[51] Bobo Hua, Ming Li, Tinggui Zhang, Chunqin Zhou, Xianqing Li-Jost, and Shao-Ming Fei. Towards Grothendieck constants and LHV models in quantum mechanics. *Journal of Physics A: Mathematical and Theoretical*, **48**, 065302, (2015). [arXiv: 1501.05507].

[52] J. L. Krivine. Constantes de Grothendieck et fonctions de type positif sur les sphères. *Advances in Mathematics*, **31**, 16–30, (1979).

[53] S. L. W. Midgley, A. J. Ferris, and M. K. Olsen. Asymmetric gaussian steering: When Alice and Bob disagree. *Phys. Rev. A*, **81**, 022101, (2010). [arXiv: 0906.4851].

[54] Vitus Händchen, Tobias Eberle, Sebastian Steinlechner, Aiko Samblowski, Torsten Franz, Reinhard F. Werner, and Roman Schnabel. Observation of one-way Einstein-Podolsky-Rosen steering. *Nat. Phot.*, **6**, 596–599, (2012). [arXiv: 1206.4446].

[55] Flavien Hirsch, Marco Túlio Quintino, Joseph Bowles, and Nicolas Brunner. Genuine hidden quantum nonlocality. *Phys. Rev. Lett.*, **111**, 160402, (2013). [arXiv: 1307.4404].

[56] Charles H. Bennett, David P. DiVincenzo, and John A. Smolin. Capacities of quantum erasure channels. *Phys. Rev. Lett.*, **78**, 3217–3220, (1997). [arXiv: quant-ph/9701015].

[57] Joseph Bowles, Flavien Hirsch, Marco Túlio Quintino, and Nicolas Brunner. Sufficient criterion for guaranteeing that a two-qubit state is unsteerable. *Phys. Rev. A*, **93**, 022121, (2016). [arXiv: 1510.06721].

[58] Kai Sun, Xiang-Jun Ye, Jin-Shi Xu, Xiao-Ye Xu, Jian-Shun Tang, Yu-Chun Wu, Jing-Ling Chen, Chuan-Feng Li, and Guang-Can Guo. Experimental quantification of asymmetric Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **116**, 160404, (2016). [arXiv: 1511.01679].

[59] Sabine Wollmann, Nathan Walk, Adam J. Bennet, Howard M. Wiseman, and Geoff J. Pryde. Observation of genuine one-way Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **116**, 160403, (2016). [arXiv: 1511.01231].

[60] Teiko Heinosaari, Jukka Kiukas, Daniel Reitzner, and Jussi Schultz. Incompatibility breaking quantum channels. *Journal of Physics A: Mathematical and Theoretical*, **48**, 435301, (2015). [arXiv: 1504.05768].

[61] Daniel Cavalcanti and Paul Skrzypczyk. Estimating measurement incompatibility via quantum steering and nonlocality. (2016). [arXiv: 1601.07450].

[62] L.A. Khalfin and B.S. Tsirelson. Quantum and quasi-classical analogs of bell inequalities. *Symposium on the Foundations of Modern Physics*, 441–460, (1985).

[63] Erika Andersson, Stephen M. Barnett, and Alain Aspect. Joint measurements of spin, operational locality, and uncertainty. *Phys. Rev. A*, **72**, 042104, (2005). [arXiv: quant-ph/0509152].

[64] Wonmin Son, Erika Andersson, Stephen M. Barnett, and M. S. Kim. Joint measurements and bell inequalities. *Phys. Rev. A*, **72**, 052116, (2005). [arXiv: quant-ph/0509133].

[65] Michael M. Wolf, David Perez-Garcia, and Carlos Fernandez. Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory. *Phys. Rev. Lett.*, **103**, 230402, (2009). [arXiv: 0905.2998].

[66] Marco Túlio Quintino, Tamás Vértesi, and Nicolas Brunner. Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality. *Phys. Rev. Lett.*, **113**, 160402, (2014). [arXiv: 1406.6976].

[67] Roope Uola, Tobias Moroder, and Otfried Gühne. Joint measurability of generalized measurements implies classicality. *Phys. Rev. Lett.*, **113**, 160403, (2014). [arXiv: 1407.2224].

[68] Mark A. Naimark. Spectral functions of a symmetric operator. *Izv. Akad. Nauk SSSR Ser. Mat.*, **4**, 277–318, (1940).

[69] Roope Uola, Costantino Budroni, Otfried Gühne, and Juha-Pekka Pellonpää. One-to-one mapping between steering and joint measurability problems. *Phys. Rev. Lett.*, **115**, 230402, (2015). [arXiv: 1507.08633].

[70] Marco Piani and John Watrous. Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **114**, 060404, (2015). [arXiv: 1406.0530].

[71] Avshalom C. Elitzur, Sandu Popescu, and Daniel Rohrlich. Quantum nonlocality for each pair in an ensemble. *Physics Letters A*, **162**, 25–28, (1992).

[72] Guifré Vidal and Rolf Tarrach. Robustness of entanglement. *Phys. Rev. A*, **59**, 141–155, (1999). [arXiv: quant-ph/9806094].

[73] HuiXian Meng, HuaiXin Cao, and WenHua Wang. The robustness of contextuality and the contextuality cost of empirical models. *SCIENCE CHINA Physics, Mechanics & Astronomy*, **59**, 640303, (2016).

[74] Michael Steiner. Generalized robustness of entanglement. *Phys. Rev. A*, **67**, 054305, (2003). [arXiv: quant-ph/0304009].

[75] K. Kraus. States, effects, and operations. fundamental notions of quantum theory. 190. Cambridge University Press, (1983).

[76] Bárbara Lopes Amaral. Emaranhamento em sistemas de dois qubits. Master's thesis, Programa de Pós-Graduação em Matemática - UFMG, (2010).

[77] Q. Y. He and M. D. Reid. Genuine multipartite Einstein-Podolsky-Rosen steering. *Phys. Rev. Lett.*, **111**, 250403, (2013). [arXiv: 1212.2270].

[78] Seiji Armstrong, Meng Wang, Run Yan Teh, Qihuang Gong, Qiongyi He, Jiri Janousek, Hans-Albert Bachor, Margaret D. Reid, and Ping Koy Lam. Multipartite Einstein-Podolsky-Rosen steering and genuine tripartite entanglement with optical networks. *Nat. Phys.*, **11**, 167–172, (2015). [arXiv: 1412.7212].

[79] D. Cavalcanti, P. Skrzypczyk, G. H Aguilar, R. V. Nery, P.H. Souto Ribeiro, and S. P. Walborn. Detection of entanglement in asymmetric quantum networks and multipartite quantum steering. *Nat. Comm.*, **6**, 7941, (2015). [arXiv: 1412.7730].

[80] Quan Quan, Huangjun Zhu, Si-Yuan Liu, Shao-Ming Fei, Heng Fan, and Wen-Li Yang. Steering Bell-diagonal states. *Scientific Reportsl*, **6**, (2016). [arXiv: 1601.00113].

[81] Tobias Moroder, Oleg Gittsovich, Marcus Huber, and Otfried Gühne. Steering bound entangled states: A counterexample to the stronger peres conjecture. *Phys. Rev. Lett.*, **113**, 050404, (2014). [arXiv: 1405.0262].

[82] Yun Zhi Law, Le Phuc Thinh, Jean-Daniel Bancal, and Valerio Scarani. Quantum randomness extraction for various levels of characterization of the devices. *Journal of Physics A: Mathematical and Theoretical*, **47**, 424028, (2014). [arXiv: 1401.4243].

[83] Bernhard Wittmann, Sven Ramelow, Fabian Steinlechner, Nathan K. Langford, Nicolas Brunner, Howard M. Wiseman, Rupert Ursin, and Anton Zeilinger. Loophole-free Einstein-Podolsky-Rosen experiment via quantum steering. *New J. Phys.*, **14**, 053030, (2012). [arXiv: 1111.0760].

[84] Jos F. Sturm. SeDuMi v1.3. CORAL Lab, Department of Industrial and Systems Engineering, Lehigh University, (2003).

[85] MOSEK ApS. The MOSEK optimization toolbox for MATLAB manual. Version 7.1. (2015).

[86] Stephen Boyd and Lieven Vandenberghe. Convex optimization. Cambridge University Press, (2004).

[87] Aharon Ben-Tal and Arkadi Nemirovski. Lectures on modern convex optimization. (2012, accessed on June 2016).

[88] Giacomo Mauro D'Ariano, Paoloplacido Lo Presti, and Paolo Perinotti. Classical randomness in quantum measurements. *Journal of Physics A: Mathematical and General*, **38**, 5979, (2005). [arXiv: quant-ph/0408115v2].

[89] MATLAB R2016a Documentation. Nonlinear optimization: `fminsearch`. (Accessed on June 2016).

[90] MATLAB R2016a Documentation. Nonlinear optimization: `fmincon`. (Accessed on June 2016).

[91] MATLAB R2016a Documentation. Nonlinear optimization: `fminunc`. (Accessed on June 2016).

[92] Daniel Cavalcanti, Leonardo Guerini, Rafael Rabelo, and Paul Skrzypczyk. General method for constructing local-hidden-variable models for multiqubit entangled states. (2015). [arXiv: 1512.00277].

[93] Flavien Hirsch, Marco Túlio Quintino, Tamás Vértesi, Matthew F. Pusey, and Nicolas Brunner. Algorithmic construction of local hidden variable models for entangled quantum states. (2015). [arXiv: 1512.00262].

[94] Michał Oszmaniec, Leonardo Guerini, Peter Wittek, Tamás Vértesi, and Antonio Acín. Simulation of general POVMs via projective measurements. Unpublished.

[95] Flavien Hirsch, Marco Túlio Quintino, Joseph Bowles, Tamás Vértesi, and Nicolas Brunner. Entanglement without hidden nonlocality. (2016). [arXiv: 1606.02215].

[96] Miguel Navascués, Stefano Pironio, and Antonio Acín. Bounding the set of quantum correlations. *Phys. Rev. Lett.*, **98**, 010401, (2007). [arXiv: quant-ph/0607119].

[97] L. Lovasz. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, **25**, 1–7, (1979).

[98] Koenraad M R Audenaert and Stefan Scheel. Quantum tomographic reconstruction with error bars: a kalman filter approach. *New Journal of Physics*, **11**, 023028, (2009). [arXiv: 0809.3359].

[99] David Gross, Yi-Kai Liu, Steven T. Flammia, Stephen Becker, and Jens Eisert. Quantum state tomography via compressed sensing. *Phys. Rev. Lett.*, **105**, 150401, (2010). [arXiv: 0909.3304].

[100] D. S. Gonçalves, C. Lavor, M. A. Gomes-Ruggiero, A. T. Cesário, R. O. Vianna, and T. O. Maciel. Quantum state tomography with incomplete data: Maximum entropy and variational quantum tomography. *Phys. Rev. A*, **87**, 052140, (2013). [arXiv: 1306.0467].

[101] A. C. Doherty, Pablo A. Parrilo, and Federico M. Spedalieri. Distinguishing separable and entangled states. *Phys. Rev. Lett.*, **88**, 187904, (2002).

[102] Cécilia Lancien, Otfried Gühne, Ritabrata Sengupta, and Marcus Huber. Relaxations of separability in multipartite systems: Semidefinite programs, witnesses and volumes. *Journal of Physics A: Mathematical and Theoretical*, **48**, 505302, (2015). [arXiv: 1504.01029].

[103] Lluis Masanes, Stefano Pironio, and Antonio Acín. Secure device-independent quantum key distribution with causally independent measurement devices. *Nat Commun*, **2**, 238, (2011). [arXiv: 1009.1567].

[104] Paul Busch. Unsharp reality and joint measurements for spin observables. *Phys. Rev. D*, **33**, 2253–2261, (1986).

[105] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde. Experimental EPR-steering using Bell-local states. *Nat Phys*, **6**, 845–849, (2010). [arXiv: 0909.0805].

[106] William K. Wootters. Optimal information transfer and real-vector-space quantum theory. (2013). [arXiv: 1301.2018].

[107] William K. Wootters. Entanglement sharing in real-vector-space quantum theory. (2015). [arXiv: 1007.1479v2].

[108] S. J. Jones and H. M. Wiseman. Nonlocality of a single photon: Paths to an Einstein-Podolsky-Rosen-steering experiment. *Phys. Rev. A*, **84**, 012110, (2011). [arXiv: 1102.5369].

[109] Roope Uola, Kimmo Luoma, Tobias Moroder, and Teiko Heinosaari. Adaptive strategy for joint measurements. (2016). [arXiv: 1604.08724].

[110] Wikipedia. The Thomson Problem. (accessed on July 2016).

[111] Stack Overflow. Fibonacci latices. (accessed on July 2016).

[112] Wolfram MathWorld. Sphere point picking. (accessed on July 2016).

[113] T. Vértesi and E. Bene. Two-qubit Bell inequality for which positive operator-valued measurements are relevant. *Phys. Rev. A*, **82**, 062115, (2010). [arXiv: 1007.2578].

[114] R. F. Werner. Steering, or maybe why Einstein did not go all the way to Bell's argument. *Journal of Physics A: Mathematical and Theoretical*, **47**, 424008, (2014).

[115] Joseph M. Renes, Robin Blume-Kohout, A. J. Scott, and Carlton M. Caves. Symmetric informationally complete quantum measurements. *J. Math. Phys.*, **45**, 2171–2180, (2004). [arXiv: quant-ph/0310075].

[116] Thomas Durt. Symmetric informationally complete POVM tomography: theory and applications. *Proceedings Symposium IEEE/LEOS Benelux Chapter, Brussels*, 215–218, (2007).

[117] G. H. Aguilar, A. Valdés-Hernández, L. Davidovich, S. P. Walborn, and P. H. Souto Ribeiro. Experimental entanglement redistribution under decoherence channels. *Phys. Rev. Lett.*, **113**, 240501, (2014). [arXiv: 1407.3996].