Universidade Federal de Minas Gerais

PHD - THESIS

Quantumness of correlations in finite dimensional systems

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In memoriam of my cousin and eternal friend Leovan Weber.

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[1] The acknowledgments is only in portuguese.

Resumo

Correlações puramente quânticas em sistemas de dimensão finita.

Tese de doutorado - Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, 2014.

Alguns fenômenos são exclusivos de sistemas quânticos, ou seja, não possuem contrapartida na mecânica clássica. Dois exemplos muito discutidos nos últimos anos são o emaranhamento e a não localidade, ambos relacionados com a existência de estados não separáveis. A superposição de estados quânticos é outra característica que merece destaque. Quando dois eventos são descritos por estados não ortogonais, o fato de estes se sobreporem implica na inexistência de um processo capaz de distingui-los. O princípio da superposição aliado ao processo de medição local em sistemas quânticos compostos resulta em uma nova classe de correlações sem contrapartida no mundo clássico e que vai além do emaranhamento. Essas correlações puramente quânticas recebem o nome de *quantumness of correlations*, e são o assunto principal desta tese. Estudamos três abordagens diferentes para as correlações puramente quânticas. Primeiramente definimos uma medida geométrica para quantificar essas correlações baseada na norma Schatten-p, a qual contém a norma do traço, norma de Hilbert-Schmidt e norma de operador. Demonstramos que essa medida de correlações é limitada inferiormente pelo emaranhamento, quando este é calculado via testemunhas de emaranhamento. A segunda abordagem das correlações puramente quânticas se deu no contexto de informação acessível e discriminação de estados quânticos. Sabe-se que, devido a superposição, estados quânticos só podem ser perfeitamente distinguidos quando são ortogonais. Sendo assim para um ensemble finito de estados quânticos existe uma quantidade máxima de informação que pode ser extraída pelo processo de medição. A quantidade de informação acessível é limitada pela cota de Holevo, e atingirá a igualdade apenas quando os estados forem ortogonais. Esse limite na quantidade de informação acessível está relacionada a incapacidade de se distinguir estados quânticos pelo processo de medição. Nosso estudo consiste em investigar a capacidade de se extrair informação, bem como em se distinguir os estados de um dado ensemble, quando restringimos o processo de medição a medições projetivas. A restrição a medição projetiva, bem como a generalização a POVMs pode ser abordada via teorema de Naimark, que atesta que uma POVM pode ser descrito como uma medição projetiva em um espaço de dimensão maior. O processo de "embeber" o estado em um espaço de dimensão maior pode ser feito, por exemplo, acoplando uma ancila ao estado. Processo este que não gera correlações entre o sistema e a ancila. Nosso principal objetivo é estudar as correlações nesse contexto para entender como elas são afetadas pelo processo de embeber o estado em um espaço de dimensão maior, uma vez que isto generaliza a forma de medição a ser realizada no sistema. Por fim estudamos correlações quânticas no contexto de partículas indistinguíveis. Obtemos uma medida de emaranhamento

para sistemas fermiônicos: a versão fermiônica da robustez generalizada de emaranhamento. Nós também introduzimos o conceito de correlações puramente quânticas para partículas indistinguíveis, obtemos quem são os estados sem correlações puramente quânticas a partir do protocolo de ativação. Como esses estados são um subconjunto dos estados separáveis, podemos garantir quais estados não são emaranhados, pois não possuem nem mesmo correlações puramente quânticas. Calculamos também uma medida dessas correlações para sistemas fermiônicos e bosônicos.
Palavras-Chave: Informação quântica, correlações quânticas, discórdia quântica, emaranhamento

Abstract

Quantumness of correlations in finite dimensional systems.

Phd Thesis - Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, 2014.

Some phenomena are exclusive of quantum systems, in other words, there no exist counterpart in classical mechanics. Two examples extensively discussed in recent years are the quantum entanglement and quantum non locality, both are related to the existence of non separable states. The superposition of quantum states is another characteristic which deserves attention. Considering two distinct events in superposition, it implies the nonexistence of a measurement process capable of discriminating them. The superposition and the local measurement process together result in a new class of correlations, without counterpart in classical world, and go beyond quantum entanglement. These quantum correlations are named *quantumness of correlations*, and they are the main issue of this thesis.

In the thesis we study three different approaches for the quantumness of correlations. Firstly, we define a geometrical measure of quantumness of correlations via the Schatten-p norm, which contain in its definition the trace norm, the Hilbert-Schmidt norm and the operator norm. We demonstrate that it is limited below by the quantum entanglement, calculated via entanglement witness. The second approach for the quantumness of correlations is in the context of accessible information and the discrimination of quantum states. It is known that quantum states only can be distinguished if they are orthogonal one each other. Then there exists a maximal amount of information which can be extracted from an ensemble of quantum states, performing measurements. The accessible information is limited by the Holevo's quantity, and the bound is attained only for orthogonal states. This limit in the amount of information that can be extracted from a quantum ensemble is related to the incapacity to distinguish quantum states by measurement process. Our study consists to investigate the capacity in to extract information, as well to distinguish the states of a given ensemble, when we are restricted to perform projective measurements. The restriction to projective measurements, as well the generalization to POVMs, can be approached via the Naimark's theorem, which state that a given POVM can be approached as a projective measurement in a embedded space. The embedding process can be performed, for example, coupling a pure ancilla on the state. Therefore this process cannot create any correlation between the system and the ancilla. The main goal is approach the quantumness of correlations in this context, to understand how they are affected by the embedding process, once that this process generalizes the measurement to be performed on the system. Finally we study quantum correlations in the context of indistinguishable particles. In our approach we obtained an entanglement measure for fermionic systems, it is a fermionic version of the generalized robustness of entanglement. We also introduced the concept of quantumness of correlations for indistinguishable particles. We calculated who are the states without quantumness of correlations from the activation protocol, in this context. As these states are a subset of separable states, we can attest what states are not entangled, once they are the states without quantumness of correlation. We also calculated a measure of quantumness of correlations for fermionic and bosonic systems.

Key-words: quantum information, quantum correlations, quantum discord, quantum entanglement

Academic Production

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Contents

Chapter 1

Introduction

In the beginning of 30 years, E. Schrodinger defines the quantum entanglement as the main characteristic of the quantum mechanics [141][1]. In the same year A. Einstein *et al.* called attention to a possible non local action between quantum systems spatially separated [57]. Only in 60 years, J. Bell explained that quantum entanglement cannot be written as a realist and local theory [13]. In 80 years A. Aspect *et al.* could measure entangled states in a quantum optics experiment [7]. During the 80 years the quantum entanglement let to be a exclusive curiosity of quantum mechanics, and started to be a scientific research topic. It was boosted by the motivation given by R. Feynman [62], he asked about the possibility to simulate quantum systems in standard computers, and proposed that there may exists an improvement in the computational cost in the opposite direction. The first protocol, following this idea, was the public quantum key protocol, known as BB84 [16]. In the next decade, the quantum information and quantum computation theory had began. The quantum entanglement could be approached as a resource for informational and computational tasks: quantum teleport [17], superdense coding [142], quantum cryptography [58] and the quantum factoring algorithm [146], are some examples.

Before the 2000 years, people considered quantum entanglement as the main difference between quantum and classical worlds, as stated by Schrodinger. However independently H. Ollivier and W. Zurek, and L. Henderson and V. Vedral found a new quantum property, without counterpart in classical systems. They named it *quantum discord*, which is related to the measurement and superposition principles of quantum mechanics in composed systems. Quantum discord is a measure of quantumness of correlations. This new kind of correlation reveals the amount of correlations destroyed during the local measurement process, and goes through the quantum entanglement. Quantumness of correlations quantifies the degree of quantumness in the correlations point of view. J. Oppeinheim *et al.* approached these kind of correlations in a thermodynamic point of view, and concluded that they are the main resource to the extraction of local pure states [117]. The existence of quantumness of correlation makes impossible to clone mixed states locally (*no-local broadcast*) [130]. It also quantifies the resource required to perform the state merging protocol [32, 106]. It bounds the amount of entanglement which two parties can increase only exchanging part of their states, and applying local operations and classical communications [35]. Quantumness of correlations is a necessary condition to the quantum computational *speed-up* [44]. Experimentally it was revealed necessary for the existence of coherent interaction between two systems [72]. These correlations are the responsible for the creation of entanglement between the system and the measurement apparatus, during the local measurement process [129, 150].

In this thesis, besides the literature review about quantumness of correlations and the compilations, of some interesting propositions and theorems in this context, we also present five original works about this issue, they were listed previously. The thesis is organized as follows. In Chap.2 we present the mathematical framework of quantum information, we revise some concepts and many results which were used in the thesis, in this chapter we also define the notation. In Chap.3

[1] A translation to English of this work can be found in the book *Quantum Theory and Measurement* in the page 152 [161].

we discuss the concept of quantum correlations. We present a revision about quantum entanglement, defining the entanglement and discussing some interesting measures of entanglement. In this chapter we introduce the concept of quantumness of correlations, discriminating who are the classical correlated states. We present some interesting results about this issue, as well a revision of the literature. Into the well known concepts there is a geometrical measure of quantumness of correlations proposed by us. In Chap.4 we discuss the relation between entanglement and quantumness of correlations. Firstly, we revise two ways to relate entanglement and quantumness of correlations: the Koash-Winter relation and the activation protocol. Then we present our contributions. We used the geometrical measure of quantumness of correlations via Schatten-p norm to calculate that it is limited bellow by the witnessed entanglement, we also discuss the relation for some particular norms and measures of entanglement [47,48]. Chapter 5 was designed to explore the embedding process in the context of local measurements. In this chapter we present a discussion about the consequences of restrict the local measurement process to projective measurements. We first discuss the concept of accessible information, which is equal to the quantum discord for classical-quantum states. Then we study the discrimination of quantum states in this context. We define a measure of *projectiveness* of a POVM. Then we explore the capacity of a POVM named *pretty good measurement* to discriminate quantum states, when a dephasing channel is performed on it, which is the process that transform a POVM in a projective measurement [49]. Our main objective of this investigation is understand what happens with geometrical measures of quantumness of correlations under embedding. It was stated that it remains unchanged, although there is no exist any formal proof of it. In Chap.6 we introduce the concept of quantumness of correlations for indistinguishable particle systems. There is no agreement about who are the separable states in these context. We approached the separable states as those which can be written as only one Slater determinant, via this definition we obtain the generalized robustness of entanglement for fermionic systems [87]. We also discuss the quantumness of correlations for indistinguishable particles, we proposed an approach via the activation protocol to obtain the class of states without quantumness of correlations. This class can be obtained from the absence of entanglement between the system and the measurement apparatus during the local measurement process. In other words, we used entanglement between distinguishable subsystems to obtain the class of states without quantumness of correlations for identical particle systems [86]. We conclude and present the future perspectives in Chap.7.

Chapter 2

Mathematical framework of quantum information

In this chapter we shall describe the mathematical framework which supports the other chapters of the thesis. We decided to detail and discuss some fundamental concepts of the quantum information theory to give comfort to the reader during the reading of the text. All of the concepts discussed in this chapter can be found in text books of quantum information and quantum mechanics [5,14,114], lectures notes [133,159,162] and the review [11] about the specific topics. If the language of quantum information is familiar to the reader, each section, even the whole chapter, can be skipped by without compromissing the next sections and/or chapters.

2.1 Density matrix

The whole thesis deal with quantum mechanics of discrete systems, and therefore the Hilbert space has finite dimension. A given Hilbert space $\mathcal{H}_N = \mathbb{C}^N$ is defined as a complex vector space, normed and with well defined inner product. For a given Hilbert space there exists a dual space $\mathcal{H}_N^*$, which is the space of linear maps from $\mathcal{H}$ to the complex numbers. For finite dimensional Hilbert spaces these two spaces are isomorphic, then $\mathcal{H}_N^* = \mathbb{C}^N$. We shall denote the space of linear transformations which act on the Hilbert space as $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$. A given linear transformation A belongs to the space $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$, if $A : \mathbb{C}^N \to \mathbb{C}^M$. If A is a square matrix it acts on the Hilbert space as: $A : \mathbb{C}^N \to \mathbb{C}^N$, then we shall denote the space where A belongs as $\mathcal{L}(\mathbb{C}^N)$. The set of linear transformations on the Hilbert space $\mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$ is also a Hilbert space, therefore it is equipped with inner product. For two operators $M, N \in \mathcal{L}(\mathbb{C}^N)$, the inner product is defined as the Hermitian form:

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B). \tag{2.1}$$

As $\mathrm{Tr}(M^\dagger N)$ is always a finite number the vector space $\mathcal{L}$ is often called the space of *bound operators*. The vector space of operators which act on the Hilbert space is also named Hilbert-Schmidt space. From the inner product, it is possible to define an Euclidean norm for $\mathcal{L}$, it is named Hilbert-Schmidt norm:

$$D_{HS}(M) = \sqrt{\mathrm{Tr}[M^\dagger M]} = |M|. \tag{2.2}$$

We shall discuss this norm and other norms for this vector space in Sec.2.3.1.

Why are we interested in a vector space composed by operators? Because it generates a vector space whose vectors have internal structure based on the mathematical properties of the operators and its matricial representation. Actually we are not indeed interested in the Hilbert-Schmidt space by itself, we are interested in some subsets of this vector space, which are characterized by the properties of the operators which belong to it. In quantum mechanics the observables are represented by operators, and the physical quantities are the expectation values of the observables, in the state that the system was prepared. In this way the eigenvalues of the observables are physical quantities. We can guarantee that a given operator A can be diagonalized, if and only if it commutes with its complex conjugate $[A, A^\dagger] = 0$. The Hermitean operators satisfy this condition, hence they have a well defined spectral decomposition. Therefore an interesting set, which belongs to the linear operators space ($\mathcal{L}$), is the set of Hermitian matrices ($\mathcal{HM}$). As the Hermitian matrices are matrices with real eigenvalues, then if we restrict this real numbers to be positive, we can find another interesting set of matrices, the set of positive matrices, denoted as $\mathcal{P}$. A given matrix P is a vector in the positive matrices vector space $\mathcal{P}(\mathbb{C}^N)$ if it satisfies the following conditions:

$$\langle\psi| P |\psi\rangle \geq 0 \quad \forall \quad \psi \in \mathbb{C}^N, \tag{2.3}$$
$$P = AA^\dagger, \tag{2.4}$$

for any vector $|\psi\rangle \in \mathbb{C}^N$ and any matrix $A \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$. The dimension of the vector space of positive matrices is equal to the dimension of the vector space of Hermitian matrices: $\dim(\mathcal{P}) = \dim(\mathcal{HM}) = N^2$. As the convex combination of positive matrices is also positive, this vector space is a convex cone in the Hermitian matrices vector space [14]. If we restrict the matrices in the positive cone to have trace=1, we arrive in another set of matrices, that is named the set of density matrices. This set also forms a vector space denoted by $\mathcal{D}$.[1] Therefore the matrices which belong to this set, or the vectors in this vector space, are named *density matrices*.

Definition 1. *A linear positive operator $\rho : \mathcal{L}(\mathbb{C}^N)$ is a density matrix and represents the state of a quantum system if it satisfies the following properties:*

- *Hermitean*
$$\rho = \rho^\dagger; \tag{2.5}$$
- *Positive semi-definite*
$$\rho \geq 0; \tag{2.6}$$
- *Trace one*
$$Tr(\rho) = 1. \tag{2.7}$$

As the convex combination of density matrices is a density matrix, the vector space $\mathcal{D}$ is a convex set whose pure states are projectors onto the real numbers. A given density matrix $\rho \in \mathcal{D}(\mathbb{C}^N)$ is a pure state if it satisfies:

$$\rho = \rho^2 \tag{2.8}$$
$$\mathrm{Tr}(\rho^3) = 1, \tag{2.9}$$

then the state ρ is a rank-1 matrix which can be written as:

$$\rho = |\psi\rangle\langle\psi|. \tag{2.10}$$

The pure states form a $2(N-1)$-dimensional subset on the (N^2-2)-dimensional boundary of $\mathcal{D}(\mathbb{C}^N)$. Every state with at least one eigenvalue equal to zero belongs to the boundary [14]. For 2-dimensional systems (it is also named qubit [143]) the boundary is just composed by pure states.

To represent a given density matrix $\rho \in \mathcal{D}(\mathbb{C}^N)$, with rank $= r$, as convex convex combination of pure states, it is necessary at least r pure states. A simplex is the minimum convex hull, therefore this minimal set of states forms a $N-1$-dimensional simplex, which is named the eigenvalue simplex, and the set of pure state is composed by the eigenvectors of ρ. The set of r eigenvectors of ρ, related to the non zero eigenvalues of ρ span a r-dimensional vector space, it is named *support*, and is denoted by $\mathrm{supp}(\rho)$. The set of eigenvectors related to the zero eigenvalues span a $N-r$

[1] The explicit demonstration of the density matrices satisfy the axioms of a vector space can be found in section 1.2 of Ref. [159].

vector space named *kernel* denoted by $\text{kern}(\rho)$. The direct sum of these two subspaces results in the Hilbert space $\mathbb{C}^N$.

The decomposition of a density matrix in pure states is not unique, for example the maximal mixture state $\mathbb{I}/N \in \mathcal{D}(\mathbb{C}^N)$ can be written as a convex combination of any set of N orthogonal states with the same coefficients, or taking all pure states with the same weight, or in many other forms. The non uniqueness of the convex combination of the density matrices is expressed in the following theorem:

Theorem 2. *A given density matrix $\rho \in \mathcal{D}(\mathbb{C}^N)$ with diagonal form:*

$$\rho = \sum_k^r \lambda_k \left|\lambda_k\right\rangle\left\langle\lambda_k\right|, \tag{2.11}$$

can be decomposed as a convex combination

$$\rho = \sum_l^M p_l \left|\phi_l\right\rangle\left\langle\phi_l\right|, \tag{2.12}$$

where $M \geq r$, if and only if there exists a unitary matrix U such that:

$$\left|\phi_l\right\rangle = \frac{1}{\sqrt{p_l}} \sum_k \sqrt{\lambda_k} U_{lk} \left|\lambda_k\right\rangle. \tag{2.13}$$

Proof. We shall prove the converse first. Multiplying $\langle\lambda_k|$ on the left of Eq.2.13, we find:

$$U_{lk} = \sqrt{\frac{p_l}{\lambda_k}} \left\langle\lambda_k|\phi_l\right\rangle. \tag{2.14}$$

These are elements of a unitary matrix because:

$$\begin{aligned}
\sum_l U^\dagger_{kl} U_{lj} &= \sum_l \frac{p_l}{\sqrt{\lambda_k \lambda_j}} \left\langle\lambda_k|\phi_l\right\rangle \left\langle\phi_l|\lambda_j\right\rangle & (2.15)\\
&= \frac{1}{\sqrt{\lambda_k\lambda_j}} \left\langle\lambda_k\right| \rho \left|\lambda_j\right\rangle & (2.16)\\
&= \delta_{kj}. & (2.17)
\end{aligned}$$

To prove the direct side of the theorem, we must check that Eq.2.13 and Eq.2.12 imply in Eq.2.11. If $\rho = \sum_k p_k \left|\phi_k\right\rangle\left\langle\phi_k\right|$ and $\left|\phi_l\right\rangle = \frac{1}{\sqrt{p_l}} \sum_k \sqrt{\lambda_k} U_{lk} \left|\lambda_k\right\rangle$, we have:

$$\begin{aligned}
\rho &= \sum_l p_l \left[\frac{1}{\sqrt{p_l}} \sum_k \sqrt{\lambda_k} U_{lk} \left|\lambda_k\right\rangle\right] \left[\frac{1}{\sqrt{p_l}} \sum_j \sqrt{\lambda_j} U^\dagger_{jl} \left\langle\lambda_j\right|\right] & (2.18)\\
&= \sum_l U^\dagger_{jl} U_{lj} \sum_{kj} \sqrt{\lambda_k\lambda_j} \left|\lambda_k\right\rangle\left\langle\lambda_j\right| & (2.19)\\
&= \sum_k^r \lambda_k \left|\lambda_k\right\rangle\left\langle\lambda_k\right|. & (2.20)
\end{aligned}$$

□

This theorem is named Schrödinger mixture, or GHJW lemma. The theorem implies that the states $|\phi_l\rangle$, which a state ρ can be decomposed convexly, are linearly dependent on the eigenvectors of ρ.

In quantum mechanics we have two ways to combine states. Consider a pure state $|\psi\rangle$, as it is a vector, it can be expanded as a linear combination of others pure states $\{|\phi_k\rangle\}_k$:

$$|\psi\rangle = \sum_k c_k |\phi_k\rangle , \tag{2.21}$$

where the coefficients c_k are complex.Their square norm $|c_k|^2$ represent the probability to give the state $|\phi_k\rangle$ performing a measurement over $|\psi\rangle$, therefore $\sum_k |c_k|^2 = 1$. Another way to decompose states is the *convex combination*.Consider state ρ, it can be written as a convex combinations of others states $\{\rho_k\}_k$:

$$\rho = \sum_k p_k \rho_k. \tag{2.22}$$

The probability vector $\vec{P} = \{p_k\}_k$ is a classical probability distribution, whose coefficients p_k represent the probability to draw the state ρ_k, as it is a convex combination $\sum_k p_k = 1$.

2.1.1 Composed systems

We we discussed above the kinematic state of the system is described by a density matrix. Now suppose the system is compose by more than only one part, then we also must compose the density matrices of the local subsystems. This composition is made via Kronecker product, or tensor product of the Hilbert spaces where the density matrices are operating on. In this way, a n-partite system is described in a Hilbert space $\mathbb{C}^N$, such that:

$$\mathbb{C}^N = \mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n}. \tag{2.23}$$

The dimension of the total Hilbert space is the product of the dimensions of the local subsystems:

$$N = \dim(\mathbb{C}_{A_1}) \times \cdots \times \dim(\mathbb{C}_{A_n}). \tag{2.24}$$

A density matrices $\rho \in \mathcal{D}(\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n})$ on this space with spectral decomposition in the form:

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| , \tag{2.25}$$

whose the eigenvectors can be decomposed in any basis in the space $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, then a given eigenstate $|\lambda\rangle$ can be written as:

$$|\lambda\rangle = \sum_{k_1} \ldots \sum_{k_n} c_{k_1,\ldots,k_n} |k_1\rangle \otimes \cdots \otimes |k_n\rangle . \tag{2.26}$$

Considering we are interested in the description of few subsystems, not in the whole system. Suppose we would like to describe the system labeled by A_1, the state of the solely subsystem is named *reduced density* and is defined via the *partial trace*:

$$\rho_{A_1} = \mathrm{Tr}_{A_2,\ldots,A_n}(\rho), \tag{2.27}$$

where the trace is taken over all the subsystem except A_1. The partial trace over some subsystem results in the density matrix of the remaining system without the information of the other parties, or the whole density matrix. As an example suppose we prepare a state composed by two atoms, the preparation consists in interact one each other during a given time. Then we separate them, and now we are just interested in one atom. The state of this atom is described by Eq.2.27.

The inverse operation of partial trace is the embedding operation. Given a system described by the density matrix $\rho_S \in \mathcal{D}(\mathbb{C}_S)$, one kind of embedding is coupling an pure ancillary system $|0\rangle_A \in \mathbb{C}_A$ on ρ_S and interacting them via a unitary operation on the global system:

$$\rho_{S,A} = U(\rho_S \otimes |0\rangle\langle 0|_A)U^\dagger, \tag{2.28}$$

where $\rho_{S,A} \in \mathcal{D}(\mathbb{C}_S \otimes \mathbb{C}_A)$. We shall discuss more about unitary operations, but it is known that unitary operation preserver the spectrum of the density matrix, therefore the embedding operation, as described above, does not change the spectrum of the density matrix, it just change the dimension of the Hilbert space. Another way to compose systems is via an operation named *purification*. We can state the purification operation as a theorem.

Theorem 3 (Purification)**.** *For any quantum state $\rho \in \mathcal{D}(\mathbb{C}_S)$ there exists a pure state $|psi\rangle_{SA} \in \mathbb{C}_S \otimes \mathbb{C}_A$ such that:*

$$\rho_S = Tr_A(|\psi\rangle\langle\psi|_{SA}). \tag{2.29}$$

Proof. Considering the spectral decomposition of ρ_S:

$$\rho_S = \sum_i \lambda_{i=1}^r |\lambda_i\rangle\langle\lambda_i|, \tag{2.30}$$

then we can write a pure state $|psi\rangle_{SA}$ such that it is decomposed in the eigenbasis of ρ_S as:

$$|psi\rangle_{SA} = \sum_{i=1}^r \sqrt{\lambda_i}\,|\lambda_i\rangle_S \otimes |i\rangle_A, \tag{2.31}$$

where the dimension of the ancillary system is equal the rank of ρ_S: $\dim(\mathbb{C}_A) = r$. It is clear that taking the trace over A the results will be the state ρ_S in Eq.2.30. □

Actually the pure state which satisfies Eq.2.29 is not unique. Given pure state in Eq.2.31 we can realize that the purification operation also does not change the spectrum of ρ_S. We also can note that the spectrum of the reduced state on $\mathbb{C}_A$, $\mathrm{Tr}_S(\rho_{SA})$, is the same of ρ_S. Indeed any bipartite pure state the reduced states have the same eigenvalues, and the decomposition like Eq.2.31 in their eigenbasis is named Schmidt decomposition. We shall discuss more about this decomposition in next chapter.

2.2 Quantum Operations

Given the definition of the operators which represent the states of the quantum system, now we need discuss the operations on these states. Considering a linear transformation $\Phi : \mathcal{L}(\mathbb{C}^N) \to \mathcal{L}(\mathbb{C}^M)$, for this map to represent a physical process, it must satisfy some conditions, determined by the physical properties of the input matrices and the outputs. Actually to describe maps acting on quantum states we have to answer the question: what are the conditions which Φ must satisfy to be a physical process? Indeed to be a physical process the transformation must map a quantum state in another quantum state $\Phi : \mathcal{D}(\mathbb{C}^N) \to \mathcal{D}(\mathbb{C}^M)$. Therefore it has to keep the properties of the state.

- Linearity: As the representation of the state as a convex combination is arbitrary, the action of the map cannot be dependent on the convex combination which the state is written. Therefore the map must be linear. For two any operators $\rho, \sigma \in \mathcal{L}(\mathbb{C}^N)$

$$\Phi(\rho + \sigma) = \Phi(\rho) + \Phi(\sigma); \tag{2.32}$$

- Positive: As a quantum state is a positive operator, the map must keep the state positive:

$$\Phi(\rho) \geq 0; \tag{2.33}$$

- Trace preserving: The eigenvalues of a given state represent a probability distribution, therefore the state after the action of a physical process must also keep this property. Therefore the quantum map has to maintain the trace of the state equal to one:

$$\mathrm{Tr}[\Phi(\rho)] = 1. \tag{2.34}$$

If a given map satisfies these three properties, it is named a *positive trace preserving* map [2]. However there is another condition which the map must satisfy. This property is related to the composition of systems in quantum mechanics. A composed system is represented by the Kronecker product,or tensor product, of the spaces[3]. Therefore, given a bipartite system described by the state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, if the map is acting just on one subsystem, the result of the action must be a quantum state. In other words the map $\mathbb{I} \otimes \Phi$ must transform states onto states. $\mathbb{I} \otimes \Phi : \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B) \to \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$. The map $\mathbb{I} \otimes \Phi$ is named a *local map*, because it acts *locally* on the composed system. If the map Φ is trace preserving, it is straightforward to check that $\mathbb{I} \otimes \Phi$ preserves the trace:

$$\mathrm{Tr}[\mathbb{I} \otimes \Phi(\rho_{AB})] = \mathrm{Tr}[\rho_A] = 1, \tag{2.35}$$

where $\rho_A = Tr_B[\rho_{AB}]$. However the positivity of the map Φ does not guarantee the positivity of the map $\mathbb{I} \otimes \Phi$. For example, suppose a state which can be written as a convex combination of product states:

$$\rho_{AB} = \sum_{k,l} p_{k,l} \rho_k^A \otimes \rho_l^B, \tag{2.36}$$

the action of the map $\mathbb{I} \otimes \Phi$, by linearity, will be:

$$\mathbb{I} \otimes \Phi(\rho_{AB}) = \sum_{k,l} p_{k,l} \rho_k^A \otimes \Phi(\rho_l^B). \tag{2.37}$$

If the map is positive $\Phi(\rho_l^B) \geq 0$ for every l, it implies that $\mathbb{I} \otimes \Phi(\rho_{AB}) \geq 0$. Nonetheless in general we cannot guarantee that. Suppose a bipartite pure state $|\psi\rangle = \sum_k c_k |a_k\rangle |b_k\rangle$, where the coefficients c_k are real numbers[4]. The map will act locally as:

$$\mathbb{I} \otimes \Phi(|\psi\rangle\langle\psi|) = \sum_{k,l} c_k c_l |a_k\rangle\langle a_l| \otimes \Phi(|b_k\rangle\langle b_l|). \tag{2.38}$$

Now suppose the transposition operation: $\Phi^T(|b_k\rangle\langle b_l|) = |b_l\rangle\langle b_k|$, which is clearly a positive and trace preserving operation. The state after the action of the partial transpose map will be:

$$\mathbb{I} \otimes \Phi^T(|\psi\rangle\langle\psi|) = \sum_{k,l} c_k c_l |a_k\rangle\langle a_l| \otimes |b_l\rangle\langle b_k| . \tag{2.39}$$

Consider the special case where $\dim(\mathbb{C}_A) = \dim(\mathbb{C}_B) = 2$ and $c_0 = c_1 = 1/\sqrt{2}$. In this case the eigenvalues of the matrix in Eq.2.39 are $\{1/2, -1/2, 0, 0\}$. This means that the matrix after the local action of the positive map Φ^T, is not a density matrix. These kind of maps are not accepted to represent physical process. In this way a given positive map may not represent a physical process. Then another condition must be imposed to a map represent a physical operation:

- Completely Positive:

$$\mathbb{I} \otimes \Phi(\rho) \geq 0. \tag{2.40}$$

  The map which satisfies this property is named *completely positive*.

The linear transformations which map quantum states in quantum states are named *quantum channels*. The space of quantum channels which maps $N \times N$ density matrices onto $M \times M$ density matrices shall be denoted in text as $\mathcal{C}(\mathbb{C}^N, \mathbb{C}^M)$.

[2] The term *linear* shall be omitted because even though it is a property we are assuming it a priori.

[3] A pedagogical explanation about the composition of quantum systems can be find on section 2.2 of Ref. [159].

[4] We shall prove in the next chapter that every bipartite pure state can be written in this way. It is named Schmidt decomposition. The tensor product of pure states shall be denoted as $|a_k\rangle \otimes |b_k\rangle = |a_k\rangle |b_k\rangle$, to simplify the notation we often also written as $|a_k\rangle \otimes |b_k\rangle = |a_k, b_k\rangle$.

Isometric and unitary operations

Before to discuss the representation of a general map, let us introduce a class of transformations which is very important for quantum mechanics: the isometric transformations.

Definition 4. *Consider a linear transformation $V : \mathbb{C}_\Gamma \to \mathbb{C}_{\Gamma'}$. If it satisfies:*

$$V^\dagger V = \mathbb{I}_\Gamma, \tag{2.41}$$

it is named an isometry.

The space of isometries $V : \mathbb{C}_\Gamma \to \mathbb{C}_{\Gamma'}$ will be denoted as $\mathcal{U}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$. Suppose a density matrix $\rho \in \mathcal{D}(\mathbb{C}_\Gamma)$, an isometry V acts on it as:

$$\rho' = V\rho V^\dagger, \tag{2.42}$$

where $\rho' \in \mathcal{D}(\mathbb{C}_{\Gamma'})$.

The set of isometries which map the space $\mathbb{C}_\Gamma$ on itself is named the set of unitary operators. If an operator $U \in \mathcal{U}(\mathbb{C}_\Gamma)$, it is named a unitary transformation and satisfies $U^\dagger U = UU^\dagger = \mathbb{I}_\Gamma$. An isometric transformation preserves the inner product, consequently the spectra of the operators. In other words, an isometry does not change the eigenvalues of the operators, nor the rank of them.

2.2.1 Quantum channel representations

Now we know what kind of maps can represent a physical process, we should define, in a concrete way, how they are represented.

Choi-Jamiolkowski representation

The Choi-Jamiolkowski representation, or Choi-Jamiolkowski isomorphism, is the representation of quantum channels as positive operators in a enlarged space. However it is not possible to represent any operator as a quantum map. Let us first define the map which formalizes this representation.

Definition 5. *Given a quantum channel $\Phi \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B)$, one defines a map $J : \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B) \to \mathcal{P}(\mathbb{C}_B \otimes \mathbb{C}_A)$ as:*

$$J(\Phi) = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|, \tag{2.43}$$

where $J(\Phi) \in \mathcal{P}(\mathbb{C}_B \otimes \mathbb{C}_A)$ is the Choi operator of the map Φ.

The Choi operator must satisfy some conditions, namely:

Theorem 6. *Given a map $\Phi \in \mathcal{T}(\mathbb{C}_A, \mathbb{C}_B)$ and its Choi operator $J(\Phi)$, the map is completely positive if and only if:*

$$J(\Phi) \in \mathcal{P}(\mathbb{C}_B \otimes \mathbb{C}_A). \tag{2.44}$$

The map is tracing preserving if and only if the following expression holds:

$$Tr_B(J(\Phi)) = \mathbb{I}_A. \tag{2.45}$$

Proof. If Φ is completely positive, then $\Phi \otimes \mathbb{I}(P) \in \mathcal{P}(\mathbb{C}_B \otimes \mathbb{C}_A)$ for any positive operator $P \in \mathcal{P}(\mathbb{C}_A)$. As $\sum_{i,j} |ii\rangle\langle jj|$ is positive, the Choi operator $J(\Phi)$ of a given completely positive map is positive. If the Choi operator $J(\Phi)$ is positive, then it can be diagonalized as:

$$J(\Phi) = \sum_i \vec{u}_i \vec{u}_i^\dagger, \tag{2.46}$$

where $\vec{u}_i \in \mathbb{C}_B \otimes \mathbb{C}_A$. We introduce the transformation Vec : $\mathcal{L}(\mathbb{C}_A, \mathbb{C}_B) \to \mathbb{C}_B \otimes \mathbb{C}_A$. Consider an operator $|i\rangle\langle j|$, applying the Vec operation[5] we obtain:

$$\text{Vec}(|i\rangle\langle j|) = |i\rangle\, |j\rangle\,. \tag{2.47}$$

Hence, there exists an operator $A_i \in \mathcal{L}(\mathbb{C}_A, \mathbb{C}_B)$, such that $\text{Vec}(A_i) = \vec{u}_i$ and

$$J(\Phi) = \sum_i \text{Vec}(A_i)\text{Vec}(A_i)^\dagger. \tag{2.48}$$

The Vec transformation satisfies the following relation:

$$\text{Vec}(A) = (A \otimes \mathbb{I}_A)\text{Vec}(\mathbb{I}_A), \tag{2.49}$$

where $\text{Vec}(\mathbb{I}_A) = \sum_i |ii\rangle$. Then the Choi operator can be written as:

$$J(\Phi) = \sum_i (A_i \otimes \mathbb{I}_A)\text{Vec}(\mathbb{I}_A)\text{Vec}(\mathbb{I}_A)^\dagger (A_i \otimes \mathbb{I}_A)^\dagger. \tag{2.50}$$

As $J(\Phi)$ is positive

$$J(\Phi) = \sum_i (A_i \otimes \mathbb{I}_A) \sum_{ij} |ii\rangle\langle jj|\, (A_i \otimes \mathbb{I}_A)^\dagger \geq 0. \tag{2.51}$$

By definition

$$J(\Phi) = \Phi \otimes \mathbb{I}_A(\sum_{i,j} |ii\rangle\langle jj|), \tag{2.52}$$

then, if the map can be represented as:

$$\Phi(P) = \sum_i A_i P A_i^\dagger, \tag{2.53}$$

the map Φ is completely positive. It is the Kraus representation.

If Φ is tracing preserving, then $\text{Tr}[\Phi(|i\rangle\langle j|)] = \delta_{i,j}$, which implies:

$$\text{Tr}_B[J(\Phi)] = \sum_{i,j} \text{Tr}[\Phi(|i\rangle\langle j|)]\, |i\rangle\langle j| = \sum_{i,j} \delta_{i,j}\, |i\rangle\langle j| = \mathbb{I}_A. \tag{2.54}$$

On the other hand, $\text{Tr}_B[J(\Phi)] = \mathbb{I}_A$, then it:

$$\sum_{i,j} \text{Tr}[\Phi(|i\rangle\langle j|)]\, |i\rangle\langle j| = \mathbb{I}_A = \sum_i |i\rangle\langle i| = \sum_{i,j} \delta_{i,j}\, |i\rangle\langle j|\,, \tag{2.55}$$

therefore $\text{Tr}[\Phi(|i\rangle\langle j|)] = \delta_{i,j}$, which means that Φ is tracing preserving. □

Usually in the literature one defines the maximally entangled state $|\phi\rangle = \frac{1}{\sqrt{|A|}} \sum_i |ii\rangle$, such that:

$$J(\Phi) = |A|\Phi \otimes (|\phi\rangle\langle\phi|), \tag{2.56}$$

where $|A| = \dim(\mathbb{C}_A)$. It is usual to omit the factor $|A|$.

Given $J(\Phi)$, it is possible to obtain the channel acting on any state $\rho \in \mathcal{D}(\mathbb{C}_A)$ with the expression:

$$\Phi(\rho) = \text{Tr}_B[J(\Phi)(\mathbb{I} \otimes \rho)], \tag{2.57}$$

where $\Phi(\rho) \in \mathcal{D}(\mathbb{C}_B)$.

As we anticipated, every quantum channel can be written as a positive operator, however not

[5] The Vec transformation is also named *Reshape*.

all positive operators represent channels. We can realize it taking the partial trace over $J(\Phi)$):

$$Tr_A[J(\Phi)] = \text{Tr}_A[\sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|] = \mathbb{I}_A. \tag{2.58}$$

As there exists positive operators whose partial trace is not the identity, or proportional to it, it is not any positive operator that represents a quantum channel.

Kraus representation

Given a quantum channel $\Phi \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B)$, there exists a set of operators $\{E_i\}_{i=1}^N$, where $E_i \in \mathcal{L}(\mathbb{C}_A, \mathbb{C}_B)$, such that the action of Φ on a state ρ can be written as:

$$\Phi(\rho) = \sum_{i=1}^{N} E_i \rho E_i^{\dagger}, \tag{2.59}$$

where $\{E_i\}_{i=1}^N$ are the *Kraus operators*. In contrast with the Choi-Jamiokowski representation, the Kraus representation is not unique. For a map Φ to be trace preserving, the Kraus operators must satisfy the following property:

Proposition 7. *The map Φ is a quantum channel, if the Kraus operators satisfy:*

$$\sum_i E_i^{\dagger} E_i = \mathbb{I}_A. \tag{2.60}$$

Proof. If the map Φ is a quantum channel it is completely positive and trace preserving. If the property in Eq.2.234 holds:

$$\text{Tr}[\Phi(\rho)] = \sum_{i=1}^{N} \text{Tr}[E_i \rho E_i^{\dagger}] = \text{Tr}[\sum_i E_i^{\dagger} E_i \rho] = \text{Tr}(\rho). \tag{2.61}$$

Now we should prove that the map is completely positive. Consider a positive operator $P \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_C)$, then:

$$(E_i \otimes \mathbb{I}_C) P (E_i \otimes \mathbb{I}_C)^{\dagger} = \sum_{\alpha} \lambda_{\alpha} (E_i \otimes \mathbb{I}_C) |\lambda_{\alpha}\rangle\langle\lambda_{\alpha}| (E_i \otimes \mathbb{I}_C)^{\dagger}, \tag{2.62}$$

where $P = \sum_{\alpha} \lambda_{\alpha} |\lambda_{\alpha}\rangle\langle\lambda_{\alpha}|$, and $\lambda_{\alpha} \geq 0$. Suppose that $(E_k \otimes \mathbb{I}_C) |\lambda_{\alpha}\rangle_{AC} = \exp(i\phi_k) |\gamma_{\alpha,k}\rangle_{BC}$. Hence as the eigenvalues of P are positive, the operator in Eq.2.62 is clearly positive. □

Stinespring representations

Consider a quantum channel $\Phi \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B)$, there exists an isometry $V \in \mathcal{U}(\mathbb{C}_A, \mathbb{C}_{BC})$ such that the quantum channel acting on a state $\rho \in \mathcal{D}(\mathbb{C}_A)$ can be represented as:

$$\Phi(\rho) = \text{Tr}_C(V \rho V^{\dagger}). \tag{2.63}$$

As the Kraus representation, the Stinespring representation is not unique. The Stinespring and the Kraus representation often are denoted as the same representation. Actually all the representations are equivalent, but the Stinespring representation is often used in the literature to prove the validity of the Kraus representation. We enunciate the equivalence between these two representations in the following proposition.

Proposition 8. *The Stinespring is equal to the Kraus representation:*

$$\Phi(\rho) = \sum_i E_i \rho E_i^{\dagger} = Tr_E(V \rho V^{\dagger}), \tag{2.64}$$

where $\{E_i\}$ are the Kraus operators, ρ is a density matrix and V is an isometry.

Proof. Consider the following isometry:

$$V = \sum_i E_i \otimes |i\rangle_C\,, \tag{2.65}$$

hence:

$$\begin{aligned} \mathrm{Tr}_E(V\rho V^\dagger) &= \mathrm{Tr}_E(\sum_i E_i \otimes |i\rangle_C\, \rho \sum_j E_j^\dagger \otimes \langle j|_C) && (2.66)\\ &= \sum_i E_i \rho E_i^\dagger. && (2.67)\end{aligned}$$

□

From this proposition it is easy to check that in the Stinespring representation the map is trace preserving and completely positive. Indeed the map is trace preserving only if the operator V is an isometry (Eq.2.65):

$$V^\dagger V = (\sum_i E_i^\dagger \otimes \langle i|_C)(\sum_j E_j \otimes |j\rangle_C) = \sum_i E_i^\dagger E_i = \mathbb{I}_A. \tag{2.68}$$

To check the positivity of the map, we can use the same argument used in Proposition 7. Consider a positive operator $P \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_B)$:

$$\Phi \otimes \mathbb{I}_B(P) = \mathrm{Tr}_C[(V \otimes \mathbb{I}_B)P(V^\dagger \otimes \mathbb{I}_B)], \tag{2.69}$$

if the isometry can be written as $V = \sum_i E_i \otimes |i\rangle_C$, it is clear that the operator $\Phi \otimes \mathbb{I}_B(P)$ remains positive.

The space labeled by C is named ancillary system or environment. The dimension of this space, for a given channel Φ, is equal to the number of Kraus operators of the channel. On the other hand, the number of Kraus operators is equal to the rank of the Choi operator $J(\Phi)$.

2.2.2 Dephasing channel

The dephasing channel $\Pi \in \mathcal{C}(\mathbb{C}^N)$ is an ideal quantum to classical channel. Because it maps as the identity every diagonal matrix, and maps all the other states as a probabilistic vectors, whose elements are the diagonal elements of the density matrix. To define a dephasing channel we have to choose the basis in the density matrix space. The diagonal operators in this basis will remains unchanged.

Definition 9. *A dephasing channel $\Pi \in \mathcal{C}(\mathbb{C}^N)$ acts on quantum states as:*

$$\Pi(\rho) = \sum_x Tr(\Pi_x \rho)\, |x\rangle\langle x|\,, \tag{2.70}$$

where $\rho \in \mathcal{D}(\mathbb{C}^N)$ and $\{\Pi_x = |x\rangle\langle x|\}_{x=1}^N$ is the basis in the space of density matrices where the dephasing acts.

The Choi operator of a dephasing channel is written as:

$$J(\Pi) = \sum_{i,j} \Pi(|i\rangle\langle j|) \otimes |i\rangle\langle j| = \sum_x |x\rangle\langle x| \otimes |x\rangle\langle x|\,. \tag{2.71}$$

As $J(\Pi)$ is a diagonal matrix with elements equal to 1, we conclude that the dephasing channel is positive, and it is clear that taking the partial trace of the Choi operator the reduced matrix is the identity, hence the dephasing channel is trace preserving.

The Kraus representation of the dephasing channel is exactly the definition 2.70:

$$\Pi(\rho) = \sum_x |x\rangle\langle x| \rho |x\rangle\langle x| = \sum_x \mathrm{Tr}(\Pi_x \rho) |x\rangle\langle x| , \tag{2.72}$$

where $\{\Pi_x = |x\rangle\langle x|\}_{x=1}^N$.

Consider a bipartite system $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, suppose that the part B will be sent through a noisy channel. The noise is represented by a dephasing channel in the computational basis $\{|x\rangle\}_{x=1}^{|B|}$, where $|B| = \dim(\mathbb{C}_B)$. By the definition the action of the channel is:

$$\mathbb{I}_A \otimes \Pi_B(\rho_{AB}) = \sum_x \mathrm{Tr}_B[\mathbb{I}_A \otimes \Pi_x \rho_{AB}] \otimes |x\rangle\langle x| , \tag{2.73}$$

where $\{\Pi_x \, |x\rangle\langle x|\}_x$. We can write $\mathrm{Tr}_B[\mathbb{I}_A \otimes \Pi_x \rho_{AB}] = p_x \rho_x^A$, then:

$$\mathbb{I}_A \otimes \Pi_B(\rho_{AB}) = \sum_x p_x \rho_x^A \otimes |x\rangle\langle x| . \tag{2.74}$$

As we expected, the state after the action of the channel is not diagonal, although it is block diagonal, in other words it can be written as:

$$\mathbb{I}_A \otimes \Pi_B(\rho_{AB}) = \begin{bmatrix} p_1 \rho_1^A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{|B|} \rho_{|B|}^A \end{bmatrix} . \tag{2.75}$$

As the dephasing channel maps the density matrix in a diagonal matrix, the reduced state B is:

$$\Pi_B(\rho_B) = \sum_i \mathrm{Tr}(\Pi_i \rho_B) |i\rangle\langle i| , \tag{2.76}$$

where $\{\Pi_i = |i\rangle\langle i|\}_i$ is the basis for the dephasing.

2.3 Norms and Distances

A distance measure between two points x and y in a vector space is a function $D(x, y)$ satisfying:

- The distance between two points is not negative:

$$D(x, y) \geq 0, \tag{2.77}$$

  it is zero when the points coincide $D(x, x) = 0$;

- It is symmetric

$$D(x, y) = D(y, x); \tag{2.78}$$

- It satisfies the triangle inequality:

$$D(x, y) \leq D(x, z) + D(z, y); \tag{2.79}$$

- It is homogeneous:

$$D(\lambda x, \lambda y) = \lambda D(x, y), \quad \forall \quad \lambda \in \mathbb{R}. \tag{2.80}$$

The function which satisfies these four conditions is a Minkowski distance. An important property of the Minkowski distance is that the corresponding unit ball created is a convex set.

The metric defined via the Minkowski distance is named *norm*. A norm of a vector $\vec{v}$ is represented as $\|\vec{v}\|$. An interesting class of norms which shall be discussed below is the Schatten-p norm.

2.3.1 Schatten-p norm

The Schatten-p norm for operators is the analogous to the l_p norm for vectors. For a given vector $\vec{v} = (v_1, ..., v_N)$ in a vector space V, the l_p norm is defined as:

$$\|\vec{v}\| = (\sum_k |v_k|^p)^{1/p}. \tag{2.81}$$

As density matrices are vectors in the space $\mathcal{D}$, it is possible to calculate the norm of these states. The analogous to the l_p norm for density matrices is named Schatten-p norm.

Definition 10. *Given a linear operator $A \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$, the Schatten-p norm is defined as:*

$$\|A\|_p = \{Tr[(AA^\dagger)^{p/2}]\}^{1/p}, \tag{2.82}$$

where $p = [1, \infty)$.

As a function of a matrix is the function of the eigenvalues, and the set of eigenvalues of a given matrix can be written as a vector, the Schatten-p norm for diagonal matrices is equal to the l_p norm. Therefore the Schatten-p norm can be written as the l_p norm of the spectral decomposition of the matrix A:

$$\|A\|_p = \left\{\sum_k |\lambda(A)_k|^p]\right\}^{1/p}, \tag{2.83}$$

where $\{\lambda(A)_k\}_k$ are the eigenvalues of A. Given this expression we realize that the Schatten norm follows a hierarchy relation for the values of p. In other words, given two norms described by $p \in [1, \infty)$ and $q \in [1, \infty)$, where $p \leq q$, they satisfy:

$$\|A\|_p \geq \|A\|_q. \tag{2.84}$$

As the Schatten norm only depends on the eigenvalues of the matrix, we realize that the norm is invariant under action of operations which keep the eigenvalues invariant. An example of this class is the isometric operations.

Proposition 11. *The Schatten-p norm is invariant under the action of isometries:*

$$\|UAV^\dagger\|_p = \|A\|_p, \tag{2.85}$$

where $U \in \mathcal{U}(\mathbb{C}_M, \mathbb{C}_K)$ and $V \in \mathcal{U}(\mathbb{C}_K, \mathbb{C}_N)$ are isometries.

Proof. Using the definition of the Schatten-p norm:

$$\begin{aligned}
\|UAV^\dagger\|_p &= \{\mathrm{Tr}[(UAV^\dagger VA^\dagger U^\dagger)^{p/2}]\}^{1/p} & (2.86)\\
&= \{\mathrm{Tr}[(U^\dagger U)^{p/2}(AA^\dagger)^{p/2}]\}^{1/p} & (2.87)\\
&= \{\mathrm{Tr}[(AA^\dagger)^{p/2}]\}^{1/p} & (2.88)\\
&= \|A\|_p. & (2.89)
\end{aligned}$$

□

As a unitary operation is an isometry, the proposition implies that the Schatten-p norm is invariant under the action of unitary operations. This property leads us to another way to define the Schatten-p norm[6]. For any norm p, there exists another norm q which respects the relation $1/p + 1/q = 1$, such that:

$$\|A\|_p = \max_{\|B\|_q \leq 1} \mathrm{Tr}(AB^\dagger), \tag{2.90}$$

where $B \in \mathcal{L}(\mathbb{C}_N, \mathbb{C}_M)$. This expression implies in the Holder inequality:

$$\mathrm{Tr}(AB^\dagger) \leq \|A\|_p \|B\|_q, \tag{2.91}$$

also for $1/p + 1/q = 1$ and any matrices $A, B \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$. The Holder inequality is a generalization for the Cauchy-Schwartz inequality.

From the Schatten-p norm, it is possible to obtain three well known norm measures:

- Operator norm ($p = \infty$):

  $$\|A\|_\infty = \max_{||u\rangle|=1} \mathrm{Tr}(A\,|u\rangle\langle u|). \tag{2.92}$$

  The infinity norm is just the largest eigenvalue of the operator.

- Hilbert-Schmidt norm ($p = 2$):

  $$\|A\|_2 = \sqrt{\mathrm{Tr}(AA^\dagger)}; \tag{2.93}$$

  The Hilbert-Schmidt norm is the sum of the square of the eigenvalues of the operator.

- Trace norm ($p = 1$):

  $$\|A\|_1 = \mathrm{Tr}(\sqrt{AA^\dagger}); \tag{2.94}$$

  The trace norm is the sum of the absolute values of the eigenvalues of the operator. The trace norm has some interesting properties for quantum information, which will be discussed below.

Trace distance

Any Schatten-p norm can be written as a measure of the distance between two operators $A, B \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^M)$.

$$\|A - B\|_p = \{\mathrm{Tr}([(A - B)(A^\dagger - B^\dagger)]^{p/2})\}^{1/p}, \tag{2.95}$$

The distance obtained via trace norm is named *trace distance*. For density matrices, the trace distance is monotonic decreasing under quantum channels.

Proposition 12. *Given two density matrices $\rho, \sigma \in \mathcal{D}(\mathbb{C}^N)$ and a quantum channel $\Phi \in \mathcal{C}(\mathbb{C}^N)$, the trace distance between the density matrices decreases under the action of a channel:*

$$\|\rho - \sigma\|_1 \geq \|\Phi(\rho - \sigma)\|_1. \tag{2.96}$$

Proof. Given the definition of the trace distance in Eq.2.90:

$$\|\Phi(\rho - \sigma)\| = \max_{\|\Lambda\|_\infty \leq 1} \mathrm{Tr}[\Lambda(\Phi[\rho - \sigma])] = \mathrm{Tr}[\tilde{\Lambda}(\Phi[\rho - \sigma])]. \tag{2.97}$$

Using the linearity of the channel and the cyclical property of the trace:

$$\|\Phi(\rho - \sigma)\| = \mathrm{Tr}[\Phi^\dagger(\tilde{\Lambda})(\rho - \sigma)]. \tag{2.98}$$

Taking again the definition Eq.2.90 and given the fact that a quantum channel is trace preserving:

$$\|\Phi(\rho - \sigma)\| = \mathrm{Tr}[\Phi^\dagger(\tilde{\Lambda})(\rho - \sigma)] \leq \max_{\|\Xi\|_\infty \leq 1} \mathrm{Tr}[\Xi(\rho - \sigma)], \tag{2.99}$$

[6] We shall not discuss here how can it be calculated, although the proof and a better discussion about this expression can be found in Chap.5 of the Ref. [21].

therefore:

$$\|\rho - \sigma\|_1 \geq \|\Phi(\rho - \sigma)\|_1. \tag{2.100}$$

□

The trace distance also can be interpreted in terms of probabilities. This comes from the fact that the trace distance between two operators can be expressed as the difference between the probability to find each operator performing a specific measurement.

Proposition 13. *Given two operators $\rho, \sigma \in \mathcal{D}(\mathbb{C}^N)$, the trace distance between them is equal to twice of the largest probability difference obtained from the same measurement:*

$$\|\rho - \sigma\|_1 = 2 \max_{0 \leq \Pi \leq \mathbb{I}} Tr[\Pi(\rho - \sigma)], \tag{2.101}$$

where the optimization is taken over all positive operators with eigenvalues bounded by one.

Proof. As the matrix $\rho - \sigma$ is Hermitian, there exists a unitary U such that:

$$\rho - \sigma = UDU^\dagger = U(D_+ - D_-)U^\dagger, \tag{2.102}$$

where $D = D_+ - D_-$ is a diagonal matrix, and $D_\pm$ are diagonal matrices with the absolute values of the positive/negative eigenvalues. If we define two other matrices $\alpha_\pm = UD_\pm U^\dagger$, we have:

$$\rho - \sigma = \alpha_+ - \alpha_-. \tag{2.103}$$

By definition the matrices α_+ and α_- spam two orthogonal spaces, therefore

$$\rho - \sigma = \alpha_+ \oplus (-\alpha_-). \tag{2.104}$$

There exists a projective measurement $\{\Pi_+, \Pi_-\}$ such that:

$$\Pi_+(\rho - \sigma)\Pi_+ = \alpha_+ \quad \text{and} \quad \Pi_-(\rho - \sigma)\Pi_- = \alpha_-. \tag{2.105}$$

Also holds from Eq.2.104:

$$|\rho - \sigma| = |\alpha_+ - \alpha_-| = \alpha_+ + \alpha_-. \tag{2.106}$$

The trace distance will be:

$$\|\rho - \sigma\|_1 = \text{Tr}(|\rho - \sigma|) = \text{Tr}(\alpha_+) + \text{Tr}(\alpha_-). \tag{2.107}$$

As $\text{Tr}(\alpha_+ - \alpha_-) = \text{Tr}(\rho - \sigma) = 0$, we have:

$$\|\rho - \sigma\|_1 = 2\text{Tr}(\alpha_+) = 2\text{Tr}(\Pi_+(\rho - \sigma)). \tag{2.108}$$

This proves that there exists an optimal measurement on Eq.2.101. For any operator $0 \leq \|\Gamma\|_\infty \leq 1$ it satisfies:

$$\begin{aligned}
2\text{Tr}(\Gamma(\rho - \sigma)) &= 2\text{Tr}(\Gamma(\alpha_+ - \alpha_-)) && (2.109)\\
&\leq 2\text{Tr}(\Gamma\alpha_+) && (2.110)\\
&\leq 2\text{Tr}(\alpha_+) && (2.111)\\
&= \|\rho - \sigma\|_1, && (2.112)
\end{aligned}$$

where we used in Eq.2.110 the positivity of Γ and in Eq.2.153 we used that its eigenvalues are less than one. □

This expression is very useful in quantum information, because it gives a statistical interpretation for the trace distance. It shall be used to prove an interesting theorem about discrimination of states in Section 2.7. The last proposition also can be obtained directly from Eq.2.90, although it is not explicit that the optimal operator is a measurement operator. Indeed the trace distance is just the inner product of the difference between the states, and the projector composed by the positive eigenstates of the difference. This proposition and Eq.2.90 were used by us to compare a geometrical measure of quantum discord and entanglement witness [47].

2.4 Measurement

Measurement is a classical statistical inference of quantum systems. The measurement process maps a quantum state in a classical probability distribution. In this section we shall discuss three different ways to define the measurement process.

2.4.1 Measurement by operators

We can define a measurement as a function $\Pi : \Sigma \to \mathcal{P}(\mathbb{C}_\Gamma)$,[7] which associates an alphabet Σ to positive operators $\{\Pi_x\}_x \subset \mathcal{P}(\mathbb{C}_\Gamma)$. The measurement process of a given density matrix $\rho \in \mathcal{D}(\mathbb{C}_\Gamma)$ is the process where an element of Σ is chosen randomly. The probability distribution which describes this random choice is described by a probability vector $\vec{p} \in \mathbb{R}^N_+$, where N is the cardinality of the random variable described by $\vec{p}$. The elements of the probability vector $\vec{p}$ are given by the expression:

$$p_x = \mathrm{Tr}(\Pi_x \rho), \tag{2.113}$$

where Π_x is the *measurement operator* associated to $x \in \Sigma$. The alphabet Σ is the set of measurement outcomes, and the vector $\vec{p}$ is the classical probability vector associated with the measurement process Π of a given density matrix ρ. As the outcomes are elements of a probability vector, these elements must be positive, which explains why the measurement operators are positive, and the sum of them must be equal to one. Therefore the measurement operators must sum to identity:

$$\sum_x \Pi_x = \mathbb{I}_\Gamma, \tag{2.114}$$

where $\mathbb{I}_\Gamma$ is the identity matrix in $\mathbb{C}_\Gamma$. It is easy to check that this condition implies in $\sum_x p_x = 1$:

$$\sum_x p_x = \sum_x \mathrm{Tr}(\Pi_x \rho) = \mathrm{Tr}(\sum_x \Pi_x \rho) = \mathrm{Tr}(\rho) = 1. \tag{2.115}$$

For instance we shall restrict the measurements to a subclass of measurement operators named *projective measurements*, which will be generalized via the Naimark's theorem in the end of the section. To obtain this kind of measurement, we should restrict the cardinality of $\vec{p}$ to be equal to at least the dimension of ρ, and the measurement operators to be projector operators. Hence they must satisfy another property:

$$\Pi_x^2 = \Pi_x, \tag{2.116}$$

for any $x \in \Sigma$. This choice implies in the following proposition:

Theorem 14. *A measurement process, represented by the set of measurement operators $\{\Pi_x\}_x$, is a projective measurement, if the set $\{\Pi_x\}_x$ must be orthogonal.*

Proof. Any set of measurement operators holds:

$$\sum_x \Pi_x = \mathbb{I} = \mathbb{I}^2 = \sum_{x,y} \Pi_x \Pi_y. \tag{2.117}$$

[7] Just to clarify the notation, when we write a subscript in the complex euclidean vector space, as $\mathbb{C}_\Gamma$, it represents a label to the space, it shall be very useful when we study composed systems. When we write a superscript in it, it represents the dimension of the complex vector space. For example, if $\dim(\mathbb{C}_\Gamma) = N$, we can also represent this space as $\mathbb{C}^N$, the usage of the notation will depend of the context.

As we are restricting to projective measurements $P_x P_x = P_x$, then:

$$\mathbb{I} = \sum_{x,y} \Pi_x \Pi_y = \sum_x \sum_{x \neq y} \Pi_x \Pi_y + \sum_x \Pi_x = \sum_x \sum_{x \neq y} \Pi_x \Pi_y + \mathbb{I}, \tag{2.118}$$

thus:

$$\sum_x \sum_{x \neq y} \Pi_x \Pi_y = 0. \tag{2.119}$$

The inner product between positive operators is non negative, which implies:

$$\mathrm{Tr}[\Pi_x \Pi_y] = 0. \tag{2.120}$$

□

Therefore the projective measurement operators form an orthonormal basis in $\mathcal{P}(\mathbb{C}_\Gamma)$. If we consider an orthonormal basis $\{|e_x\rangle\}$, where the vectors $|e_x\rangle$ span $\mathbb{C}_\Gamma$, this set represents a projective measurement for $\Pi_x = |e_x\rangle\langle e_x|$. This measurement defines a convex hull in $\mathcal{P}(\mathbb{C}_\Gamma)$, where a measured state represents a pure state in this convex hull. The post-measurement state is described by the expression:

$$\rho_x = \frac{\Pi_x \rho \Pi_x}{\mathrm{Tr}(\Pi_x \rho)}. \tag{2.121}$$

An example of projective measurement is the well known Stern-Gerlach experiment.

2.4.2 Measurement as a quantum channel

As physical processes are described by quantum channels, we can describe the classical statistical inference of the quantum measurements as a channel, with the output as classical registers. A channel which maps a quantum state in a classical probability distribution is the dephasing channel. In general the definition of a channel $\Phi \in \mathcal{C}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$, which map a quantum state to a classical probability distribution must satisfy:

$$\Phi = \Pi \Phi, \tag{2.122}$$

where $\Pi \in \mathcal{C}(\mathbb{C}_{\Gamma'})$. Given this definition, we are able to write explicitly how is the action of the measurement channel, or measurement map.

Theorem 15. *A given map $\Phi \in \mathcal{C}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ is a measurement if and only if:*

$$\Phi(\rho) = \sum_x Tr(M_x \rho) |e_x\rangle\langle e_x|, \tag{2.123}$$

where $\rho \in \mathcal{D}(\mathbb{C}_\Gamma)$, $M_x \in \mathcal{P}(\mathbb{C}_\Gamma)$ and $|e_x\rangle \in \mathbb{C}_{\Gamma'}$.

Proof. If $\Pi \circ \Phi(\rho) = \Phi(\rho)$ and it is a channel, then applying a dephasing channel on $\Phi(\rho)$:

$$\begin{aligned} \Pi \circ \Phi(\rho) &= \sum_x \mathrm{Tr}[\Phi(\rho)\Pi_x] |e_x\rangle\langle e_x| && (2.124) \\ &= \sum_x \mathrm{Tr}[\Phi^\dagger(\Pi_x)\rho] |e_x\rangle\langle e_x| && (2.125) \\ & && (2.126) \end{aligned}$$

where $\Pi_x = |e_x\rangle\langle e_x| \in \mathcal{P}(\mathbb{C}_{\Gamma'})$. If we named $\Phi^\dagger = M_x$, then:

$$\Phi(\rho) = \sum_x \mathrm{Tr}[M_x \rho] |e_x\rangle\langle e_x|. \tag{2.127}$$

For the set of positive operators $\{M_x\}_x$ to be a set of measurement operators, the sum of the operators must be equal to one:

$$\sum_x M_x = \sum_x \Phi(\Pi_x)^\dagger = \Phi(\mathbb{I}_{\Gamma'})^\dagger, \tag{2.128}$$

as we are assuming that Φ is a channel it is CPTP:

$$\Phi(\mathbb{I}_{\Gamma'})^\dagger = \sum_k \left(E_k \mathbb{I}_{\Gamma'} E_k^\dagger \right)^\dagger = \sum_k E_k^\dagger E_k = \mathbb{I}_\Gamma, \tag{2.129}$$

where $\{E_k\}$ are the Kraus operators of the map Φ, which implies:

$$\sum_x M_x = \mathbb{I}_\Gamma. \tag{2.130}$$

To prove the converse part we should prove two things:

1 The set $\{M_x\}_x$ is unique;

2 Given a set $\{M_x\}_x$, Φ is a channel.

To prove the Item 1 we assume the existence of another set $\{N_x\}_x$ for Φ, therefore:

$$\Phi(\rho) = \sum_x \mathrm{Tr}[N_x \rho] \left| e_x \right\rangle \left\langle e_x \right|, \tag{2.131}$$

as the map is the same:

$$\sum_x \mathrm{Tr}[N_x \rho] \left| e_x \right\rangle \left\langle e_x \right| - \sum_x \mathrm{Tr}[M_x \rho] \left| e_x \right\rangle \left\langle e_x \right| = 0 \tag{2.132}$$

$$\sum_x \mathrm{Tr}[(N_x - M_x)\rho] \left| e_x \right\rangle \left\langle e_x \right| = 0 \tag{2.133}$$

$$N_x - M_x = 0. \tag{2.134}$$

The item 2 can be proved via the Choi representation of Φ:

$$J(\Phi) = \sum_x \Phi(\left| e_x \right\rangle \left\langle e_x \right|) \otimes \left| e_x \right\rangle \left\langle e_x \right| = \sum_x M_x^\dagger \otimes \left| e_x \right\rangle \left\langle e_x \right| \geq 0, \tag{2.135}$$

which implies that the map is completely positive. As $J(\Phi) \in \mathcal{L}(\mathbb{C}_\Gamma \otimes \mathbb{C}_E)$, the tracing over the subsystem E results in:

$$\mathrm{Tr}_E[J(\Phi)] = \sum_x M_x = \mathbb{I}_\Gamma, \tag{2.136}$$

that is the trace preserving condition. Therefore the map Φ is CPTP, which implies that it is a quantum channel. □

In order to differ the set of measurement channels from general quantum channels, we shall represent the former as $\mathcal{P}$. A given measurement map $\mathcal{M} \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ is a quantum channel which maps a density matrix in a probability vector, $\mathcal{M} : \mathcal{D}(\mathbb{C}_{\Gamma'}) \to \mathbb{R}^+_{\Gamma'}$. This probability vector is described by a diagonal density matrix in Eq.2.140. The dimension of $\mathbb{C}_{\Gamma'}$ is the number of outcomes of the measurement represented by $\mathcal{M}$. In general the measurement operators will be consider rank-1, because we can always diagonalize them and describe the same measurement as a measurement with more outcomes. Actually we are interested in measurements described by POVMs, whose elements are rank-1 and linearly independent. For this case the number of elements of the POVM is at most N^2, where N is the dimension of the density operators vector space. If the measurement is projective, the number of outcomes is equal to the dimension of the system, then a projective measurement map Π will be represented as $\Pi \in \mathcal{P}(\mathbb{C}_\Gamma)$.

With the measurement channel Φ is possible to recover the description of the last section where $\{M_x\}_x$ is the set of measurement operators, therefore $\mathrm{Tr}[M_x\rho] = p_x$ are the elements of the probability vector which represents the post-measurement state. Thuswe can write the measurement map as:

$$\Phi(\rho) = \sum_x p_x |e_x\rangle\langle e_x| \,. \tag{2.137}$$

For the case where the number of outcomes is at least the dimension of the density matrix and the operators correspond to a projective measurement $\{\Pi_x\}_x$, the channel is a dephasing in the basis $\{\Pi_x = |e_x\rangle\langle e_x|\}_x$. Its Kraus representation is simply

$$\Phi_\Pi(\rho) = \sum_x \Pi_x\rho\Pi_x. \tag{2.138}$$

Via the Naimark's theorem it is possible to obtain any measurement channel from a dephasing channel on a state in a enlarged space. It will be study in the next section.

Given the Kraus representation of the dephasing channel in Eq.2.138, it is possible to obtain the Stinerspring representation of the measurement channel. Let us enunciate the following theorem:

Theorem 16. *Given a projective measurement channel $\Pi \in \mathcal{P}(\mathbb{C}_\Gamma)$ there exists an isometry $V \in \mathcal{U}(\mathbb{C}_\Gamma, \mathbb{C}_\Gamma \otimes \mathbb{C}_E)$ such that:*

$$Tr_E[V\rho V^\dagger] = \sum_x \Pi_x\rho\Pi_x, \tag{2.139}$$

where $\{\Pi_x\}_x \subset \mathcal{P}(\mathbb{C}_\Gamma)$ are the projective measurement operators.

Proof. If the measurement operators are projectors $\Pi_x = |e_x\rangle\langle e_x|$, then we can choose an isometry which acts as:

$$V |e_x\rangle_\Gamma = |e_x\rangle_\Gamma |e_x\rangle_E \,. \tag{2.140}$$

The action of V on ρ results in:

$$V\rho V^\dagger = \sum_{i,j} c_{ij} |e_i\rangle\langle e_j|_\Gamma \otimes |e_i\rangle\langle e_j|_E \,, \tag{2.141}$$

tracing over E:

$$\mathrm{Tr}_E[V\rho V^\dagger] = \sum_i c_{ii} |e_i\rangle\langle e_i|_\Gamma = \sum_i \Pi_i\rho\Pi_i. \tag{2.142}$$

□

The isometry V can be written without loss of generality as:

$$V = U(\mathbb{I} \otimes |0\rangle), \tag{2.143}$$

where $U \in \mathcal{U}(\mathbb{C}_\Gamma \otimes \mathbb{C}_E)$ is a unitary operation, such that $U |e_k\rangle |0\rangle = |e_k\rangle |e_k\rangle$. Therefore the measurement map on Γ can be described as a unitary evolution between the system and a ancilla E, followed by a dephasing on the ancilla:

$$\Phi_\Pi(\rho) = \mathrm{Tr}_E[U(\rho_\Gamma \otimes |0\rangle\langle 0|_E)U^\dagger] = \sum_x \mathrm{Tr}_E[(\mathbb{I} \otimes \Pi_x)U(\rho_\Gamma \otimes |0\rangle\langle 0|_E)U^\dagger(\mathbb{I} \otimes \Pi_x)]_{\Pi_x}, \tag{2.144}$$

where the trace is taken in the basis $\{|e_x\rangle\}_x$, then $\Pi_x = |e_x\rangle\langle e_x|$. In this way we can interpret the Stinerspring representation of the measurement map as: the system is interacting with the measurement apparatus, described by the ancillary system E, the interaction with the measurement apparatus and tracing it over result in a new state which represents the post-measured state.

2.4.3 Local measurements

As the measurement can be described by a quantum channel, we can study how quantum measurements can be applied locally.

Definition 17. *Given a N-partite composed system, represented by the state* $\rho_{A_1,\ldots,A_N} \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_N})$*, we define the measurement applied locally on each subsystem:*

$$\Phi_{A_1} \otimes \cdots \otimes \Phi_{A_N}(\rho_{A_1,\ldots,A_N}) = \sum_{\vec{k}} Tr[M^{A_1}_{k_1} \otimes \cdots \otimes M^{A_N}_{k_N} \rho_{A_1,\ldots,A_N}] \left|\vec{k}\right\rangle\left\langle\vec{k}\right|, \tag{2.145}$$

where $\left|\vec{k}\right\rangle = |k_1\rangle \otimes \cdots \otimes |k_N\rangle$ *and the label* $\vec{k}$ *in the sum represents the set of indexes* $k_1, \ldots, k_N$. $\{M^{A_x}_{k_x}\}_{k_x}$ *are measurement operators on each subsystem.*

Suppose the measurement is applied on some subsystems, remaining other subsystems unmeasured. As the identity operation is a projector, it is possible to represent the unmeasured subsystems substituting the measurement operators on these systems by the identity operation, and the trace is taken partially on the measured systems. For example, consider a bipartite system $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, and suppose the measurement is applied on the system B, then the measurement map will be written as:

$$\mathbb{I}_A \otimes \Phi_B(\rho_{AB}) = \sum_x \mathrm{Tr}_B[\mathbb{I}_A \otimes M^B_x \rho_{AB}] \otimes |b_x\rangle\langle b_x|\,. \tag{2.146}$$

As the measurement is not applied on A, the post-measured state on A will remain the same. If we write $p_x = \mathrm{Tr}_{AB}[\mathbb{I}_A \otimes M^B_x \rho_{AB}]$ and $\rho^A_x = \frac{\mathrm{Tr}_B[\mathbb{I}_A \otimes M^B_x \rho_{AB}]}{\mathrm{Tr}_{AB}[\mathbb{I}_A \otimes M^B_x \rho_{AB}]}$, the post-measured state will be written as:

$$\mathbb{I}_A \otimes \Phi_B(\rho_{AB}) = \sum_x p_x \rho^A_x \otimes |b_x\rangle\langle b_x|\,. \tag{2.147}$$

The local measurement is also a statistical inference process, even though the global post-measured state is not a probability vector, however as the measurement is applied locally it characterizes a local statistical inference process which implies that the local post-measured state is a probability vector. As the measurement process is a statistical inference, in the local measurement process the purely quantum features, of the composed system, are lost during the local measurement process. As we shall discuss in the last chapters, the local measurement process destroys the quantum correlations between the systems. Indeed the post-measured state is not a classical probability distribution, although it only has classical correlations.

2.4.4 Naimark's theorem

It is obvious that the expectation values of projective measurement operators result in elements of a probability vector for the elements of the diagonal of the density matrix, which by definition sum to one. However for measurements in general, where the measurement operators are POVM elements, it is imposed that the sum of the operators is equal to identity, which implies that the expectation value of the operators are elements of a probability vector. On the other hand, for POVMs whose elements are rank-1 and linearly independent, it is possible to associate a projective measurement on an enlarged space. This result is named Naimark's theorem.

Theorem 18 (Naimark's Theorem)**.** *Given a quantum measurement* $\mathcal{M} \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$*, with POVM elements* $\{M_x\}_{x=0}^M$*, there exists a projective measurement* $\Pi \in \mathcal{P}(\mathbb{C}_\Gamma)$*, with elements* $\{\Pi_y\}_{y=0}^M$ *such that:*

$$Tr(M_x \rho) = Tr(\Pi_x V \rho V^\dagger), \tag{2.148}$$

where $V \in \mathcal{U}(\mathbb{C}_{\Gamma'}, \mathbb{C}_{\Gamma'}$ *is an isometry.*

Proof. The expectation value of Π_x in the state $V\rho V^\dagger$ is:

$$\mathrm{Tr}(\Pi_x V \rho V^\dagger) = \mathrm{Tr}(V^\dagger \Pi_x V \rho) = \mathrm{Tr}(M_x \rho). \tag{2.149}$$

We must check that $\{M_x\}_x$ is a set of measurement operators. If $\{\Pi_x\}_x$ are the measurement operators of a projective measurement, the sum of them must be equal to the identity $\sum_x \Pi_x = \mathbb{I}_{\Gamma'}$.

This implies:

$$\sum_x M_x = \sum_x V^\dagger \Pi_x V = V^\dagger V = \mathbb{I}_\Gamma. \tag{2.150}$$

Which completes the proof. □

The action of the isometry on the state ρ in the Naimark's theorem is named embedding operations. The simplest way to embed the state ρ is coupling a pure ancilla. In this way the isometry will be $V = \mathbb{I}_\Gamma \otimes |0\rangle_E$ and the enlarged space $\mathbb{C}_{\Gamma'} = \mathbb{C}_\Gamma \otimes \mathbb{C}_E$. For this simple case the relation between the POVM elements $\{M_x\}_x$ and the projective measurement on the enlarged space $\{\Pi_x\}_x$ is just:

$$M_x = (\mathbb{I}_\Gamma \otimes \langle 0|_E)\Pi_x(\mathbb{I}_\Gamma \otimes |0\rangle_E). \tag{2.151}$$

In chapter 5 we shall study the consequences of the embedding in the context of local measurements, and that is possible to extract more information applying a projective measurement on the embedded state.

2.5 Semidefinite programing

Semidefinite programming (SDP) is a very useful computational and analytic tool for optimization over convex sets. In this section we shall define the SDP optimization problems for Hermitean matrices.

Definition 19. *Given two operators $A \in \mathcal{HM}(\mathbb{C}_A)$ and $B \in \mathcal{HM}(\mathbb{C}_B)$ and a map $\Phi \in \mathcal{T}(\mathbb{C}_A, \mathbb{C}_B)$, which preserver the Hermiticity, semidefinite programming is defined by the triple (Φ, A, B) with two optimization problems:*

$$\begin{array}{lclclc} & \textit{Primal Problem} & & & \textit{Dual Problem} \\ \textit{maximize}: & Tr(AX) & & \textit{minimize}: & Tr(BY) \\ \textit{Subject to:} & \Phi(X) = B & & \textit{Subject to:} & \Phi^\dagger(Y) \geq A \\ & X \in \mathcal{P}(\mathbb{C}_A) & & & Y \in \mathcal{P}(\mathbb{C}_B) \end{array} \tag{2.152}$$

The solution of the primal and dual problems are given by the following optimization problems:

$$\alpha = \sup_{X \in \mathcal{A}} Tr(AX), \tag{2.153}$$

$$\beta = \inf_{X \in \mathcal{B}} Tr(BY), \tag{2.154}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the primal and dual feasible sets respectively. The feasible sets are defined as the set of possible solutions for the primal and dual problems:

$$\mathcal{A} = \{X \in \mathcal{P}(\mathbb{C}_A) : \Phi(X) = B\}, \tag{2.155}$$

$$\mathcal{B} = \{Y \in \mathcal{P}(\mathbb{C}_B) : \Phi^\dagger(Y) \geq A\}. \tag{2.156}$$

The feasible sets will be empty sets if and only if the solutions are $\alpha = -\infty$ and $\beta = \infty$. It is important to note that the values of α and β may not be achieved by any value of X and Y, in the feasible domain.

The dual problem is concave even the primal problem is not convex, therefore the dual optimization problem is a *convex optimization*. A convex optimization problem is a minimization problem where the function is convex, and the feasible set is a convex set. The solution for the dual problem β is the best lower bound for the solution of the primal problem α, this lower bound is obtained from the dual convex optimization problem. This relation between the primal and dual problem is named *duality*.

Theorem 20 (Weak duality)**.** *Given a semidefinite programming (Φ, A, B) and let α and β definite in Eq.2.153, the weak duality holds:*

$$\alpha \geq \beta. \tag{2.157}$$

Proof. If the theorem is valid for any operator $X \in \mathcal{P}(\mathbb{C}_A)$ and $Y \in \mathcal{P}(\mathbb{C}_B)$, it holds for the optimal values. Assuming that the solution is in the feasible sets Eq.2.155, it implies that the primal problem holds:

$$\mathrm{Tr}(AX) \geq \mathrm{Tr}(\Phi(Y)X), \tag{2.158}$$

and the dual problem:

$$\mathrm{Tr}(BY) \leq \mathrm{Tr}(\Phi^{\dagger}(X)Y). \tag{2.159}$$

As $\mathrm{Tr}(\Phi(Y)X) = \mathrm{Tr}(\Phi^{\dagger}(X)Y)$ it proves the theorem. □

The difference between the optimal primal point and optimal dual point: $\alpha - \beta$ is named optimal dual gap. If the optimal dual gap is zero, i.e $\alpha = \beta$, it is named strong duality. As we are assuming in Eq.2.152 that the primal problem is also a convex optimization problem, the strong duality usually holds. However to it really holds, we need impose other conditions. The other conditions are named Slater's conditions, which we shall enunciate as a theorem[8].

Theorem 21 (Slater's conditions)**.** *Given the semidefinite programming defined by the triple (Φ, A, B), the feasible sets $\mathcal{A}$ and $\mathcal{B}$ defined in Eq.2.153 and the optimal solutions α and β defined in Eq.2.155. If the semidefinite programming satisfies:*

$$\begin{array}{cccccc} & \textit{Primal Problem} & & & \textit{Dual Problem} \\ \textit{maximize}: & Tr(AX) & & \textit{minimize}: & Tr(BY) \\ \textit{Subject to:} & \Phi(X) = B & & \textit{Subject to:} & \Phi^{\dagger}(X) > A \\ & X \in \mathcal{P}(\mathbb{C}_A) & & & Y \in \mathcal{P}(\mathbb{C}_B) \end{array} \tag{2.160}$$

then $\alpha = \beta$ and there exist a $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ such that $Tr(AX) = \alpha$ and $Tr(BY) = \beta$.

In the cases which the semidefinite programming is used in this text, the strong duality is valid. We shall use semidefinite programming to prove a theorem in the next section about quantum state discrimination. SDP will also be used in the context of entanglement witness.

2.6 Classical and quantum information

In this section we shall discuss the classical and quantum information theory. The focus of this section is the definition of the measures of information for classical and quantum systems (entropy). We also prove some propositions about the entropies which will be used throughout the text.

2.6.1 Shannon entropy

Consider a random variable X. Each realization x of X belongs to an alphabet $\mathcal{X}$[9]. Let p_x represent the probability of x to occur. Each p_x is an element of a probability vector $\vec{P}$, which represents the probability density function of X. The information function $i(x)$ quantifies the surprise that we have after obtaining a result x, measuring X randomly. It can be defined as:

$$i(x) = -\log_2(p_x), \tag{2.161}$$

[8] The proof of this theorem can be found on Section 20.2 of reference [159].

[9] As we are interested in quantum systems with finite dimensions, we are only considering discrete probability distributions, where the number of possible results is finite.

where the log function in the basis two measures the surprise in bits. Actually to learn about X, we need to measure it many times, the average of the information in bits is an entropy function named *Shannon entropy*.

Definition 22 (Shannon entropy)**.** *Let the probability vector $\vec{P} = \{p_x\}_x$ describe the probability density function of a random variable X. The Shannon entropy $H(X)$ for the random variable X is defined as the average of the information in Eq.2.161:*

$$H(X) = -\sum_x p_x \log_2(p_x). \tag{2.162}$$

By convention $0 \log_2 0 = 0$.

The Shannon entropy intrinsically has an operational interpretation. Suppose two experimentalists Alice and Bob. Alice should send a message to Bob. She will codify an alphabet $\mathcal{X}$ in a random variable X. Suppose this message is codified in bits, then the Shannon entropy measures the amount of bits, which Alice needs to codify her message. As we are always interested in correlation between systems, one interesting situation is when we have two random variables (X, Y). Supposing, for example, that Alice sends a message codified in the random variable X to Bob, via a noise channel, the noise will disturb the message and Bob will obtain a different probability distribution Y. They would like to know how correlated are the message sent by Alice and the message received by Bob. In other words, they must quantify how much information Bob learns about X knowing Y. The joint probability distribution (X, Y) can be represented by a probability vector with elements $p(x, y)$. The joint probability represents the intersection between the probability distributions X and Y, in the space of probabilities. The marginal distributions $p_x = \sum_y p(x, y)$ and $p(y) = \sum_x p(x, y)$ represent the elements of the probability vectors, for the random variables X and Y, respectively. It is possible to define a function which quantifies how distinct are the random variables X and Y, It is named *relative entropy* or Kullback-Leibler entropy.

Definition 23 (Relative entropy)**.** *Consider two random variables X and Y given by the probability vectors with elements $\{p_x\}$ and $\{p(y)\}$ respectively. The distance between these probability distributions is represent by the relative entropy and written as:*

$$D(X||Y) = \sum_{x,y} p_x \log_2(\frac{p_x}{p(y)}). \tag{2.163}$$

This function is always, positive and zero only for $X = Y$.

One can see that the relative entropy is not symmetric and $D(X||Y)$ can be different from $D(Y||X)$. Indeed the relative entropy is not a measure of distance in the sense of 2.3.1, because it does not measure the difference between the probability distributions, it is a distinguishability measure.

As the joint probability is the intersection between two probabilities distributions, one can be interested in quantifying, in bits, the amount of information that they have in common. The quantity which measures in bits the amount of correlations between two probability distributions is named *mutual information*. It is defined as the difference between the information which the probability distributions have separately, with the information which they have in common.

Definition 24 (Mutual information)**.** *Given two random variables X and Y, the joint probability distribution is represented as the probability vector with elements $\{p(x, y)\}_{x,y}$, the amount of bits of information which the probability distributions have in common is quantified by the function:*

$$H(X : Y) = H(X) + H(Y) - H(X, Y), \tag{2.164}$$

where $H(A) = \sum_a p_a \log_2(p_a)$ is the Shannon entropy of the random variable A.

The mutual information also can be interpreted as how much distinct the joint probability $\{p(x,y)\}_{x,y}$ is from the product of the distributions $\{p_x p(y)\}$. This interpretation can be recovered from the following proposition.

Proposition 25. *Given two random variables X and Y, the mutual information of them can be written as:*

$$H(X:Y) = D((X,Y)||X \cdot Y) = \sum_{x,y} p(x,y)\log_2[\frac{p(x,y)}{p_x p(y)}].,\tag{2.165}$$

where $\{p(x,y)\}_{x,y}$ *are the elements of the joint probability vector,* $\{p_x\}_x$ *and* $\{p_(y)\}_y$ *are the elements of the marginal probability distributions.*

Proof. Given the definition of mutual information and Shannon entropy, we can write the mutual information as:

$$H(X:Y) = -\sum_x p_x \log_2 p_x - \sum_y p(y)\log_2 p(y) + \sum_{x,y} p(x,y)\log_2 p(x,y),\tag{2.166}$$

using the marginal probabilities definition $p_x = \sum_y p(x,y)$ and $p(y) = \sum_x p(x,y)$:

$$\begin{aligned} H(X:Y) &= \sum_{x,y} p(x,y)\{-\log_2 p_x - \log_2 p(y) + \log_2 p(x,y)\} &(2.167)\\ &= \sum_{x,y} p(x,y)\left\{\log_2 \frac{p(x,y)}{p_x p(y)}\right\} &(2.168)\\ &= D((X,Y)||X\cdot Y). &(2.169)\end{aligned}$$

□

Suppose, again, Alice is sending a message for Bob via a noisy channel, the probability of Bob obtaining the result $Y=y$ if Alice sent the value $X=x$ is named conditional probability $p(y|x)$, and can be obtained from the joint probability and the marginal probability:

$$p(y|x) = \frac{p(x,y)}{p_x}.\tag{2.170}$$

If the probability distributions are uncorrelated, which means that $p(x,y) = p_x p(y)$, the conditional probability of Bob measuring Y is independent of Alice, therefore $p(y|x) = p(y)$. The probability of Bob guessing a result $Y=y$, considering that he had the output $X=x$, is the conditional probability $p(x|y) = \frac{p(x,y)}{p(y)}$. As the joint probability is symmetric we obtain:

$$p(y)p(x|y) = p_x p(y|x).\tag{2.171}$$

This expression is named Bayes rule.

The mutual information measures the amount of information we have about X given that we know Y. Then we should define the entropy of X when Y is known. It is named *conditional entropy*.

Definition 26 (Conditional entropy)**.** *Given two random variables X and Y represented by the probability vector with elements* $\{p(x,y)\}_{x,y}$*, the uncertainty about X when Y is known is quantified by the function:*

$$H(X|Y) = H(X,Y) - H(Y) = \sum_{x,y} p(x,y)\log_2(p(x|y)).\tag{2.172}$$

The mutual information can be written in function of the conditional probability:

$$H(X:Y) = H(X) - H(X|Y) = H(Y) - H(Y|X).\tag{2.173}$$

From this expression we can interpret the mutual information as the amount of information that is obtained about X when Y is known, and *vice versa*.

2.6.2 von Neumann entropy

For information to be exchanged, it must be carried by a physical system. Therefore classical information always can be processed to be carried by quantum systems. In other words, one can prepare an ensemble of quantum states $\xi = \{p_x, \rho_x\}_x$ according to some random variable X. Classical information can be extracted from a quantum ensemble, in the form of a variable Y performing measurements on the quantum system.

The conditional probability distribution to obtain a value y, given the input the state ρ_x is:

$$p(y|x) = \mathrm{Tr}(M_y \rho_x), \tag{2.174}$$

where $\{M_y\}_y$ is the set of measurement operators. The joint probability distribution X and Y is given by:

$$p(x,y) = p_x \mathrm{Tr}(M_y \rho_x). \tag{2.175}$$

The probability distribution of Y is obtained from the marginal probability distribution:

$$p(y) = \sum_x p(x,y) = \sum_x p_x \mathrm{Tr}(M_y \rho_x) = \mathrm{Tr}(M_y \sum_x p_x \rho_x). \tag{2.176}$$

Considering the Bayes rule:

$$p(x,y) = p_x p(y|x) = p(y) P(x|y), \tag{2.177}$$

it is possible to obtain the conditional probability distribution with elements:

$$P(x|y) = \frac{p_x p(y|x)}{p(y)}. \tag{2.178}$$

Even in the case we are preparing always a given system in the same state, there exists an uncertainty about the measurement of an observable. This uncertainty comes from the fact that, even in the case which the system is always prepared in the same state, we are restrict to obtain expectation values, which measures the average of results. The probability distributions presented above are evidencing this uncertainty, when the observables which are being measured compose a set of POVM elements. These probability distributions are classical probability distributions extracted from quantum systems, and the Shannon entropy quantifies the degree of surprise related to a given measurement result.

It is also possible to define a quantum analogous to the Shannon entropy. But as the quantum systems posses both quantum and classical uncertainties. We also expect that the entropy is only dependent of the density matrix, as in the classical case it only depends of the probability distribution. The quantum entropy is named von Neumann entropy and, in analogy with the Shannon entropy, is defined as the expectation value of the operator $\log_2(\rho)$, which measures the information, as the function in Eq.2.161.

Definition 27 (von Neumann entropy)**.** *Given a density operator $\rho \in \mathcal{D}(\mathbb{C}^N)$, the quantum version of the Shannon entropy is defined by the function:*

$$S(\rho) = -Tr[\rho \log_2 \rho]. \tag{2.179}$$

The von Neumann entropy can be rewritten as:

$$S(\rho) = -\sum_k \lambda_k \log(\lambda_k), \tag{2.180}$$

where $\{\lambda_k\}_k$ for $\rho = \sum_k \lambda_k |k\rangle\langle k|$. The von Neumann entropy has the same interpretation of the Shannon entropy for the probability distribution composed by the eigenvalues of the density matrix, in other words, it measures the uncertainty about the eigenvalues of the density matrix in qubits. The von Neumann entropy is zero if the state is pure, and it is maximum if the state is the maximally mixed state $\mathbb{I}/N$, where it is $S(\mathbb{I}/N) = \log_2 N$. As the von Neumann entropy only depends on the spectral decomposition of the state, one realizes that it is invariant under the action of isometries:

$$S(\rho) = S(V\rho V^\dagger), \tag{2.181}$$

where $V \in \mathcal{U}(\mathbb{C}^N, \mathbb{C}^{N'})$ is an isometry.

For composed systems the von Neumann entropy is analogous to the Shannon entropy for the joint probability. For a bipartite state ρ_{AB}, the joint von Neumann entropy is:

$$S(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB} \log_2 \rho_{AB}). \tag{2.182}$$

Until now we have seen that the properties of the von Neumann entropy are not so different from the Shannon entropy. However as the $\mathrm{supp}(\rho_{AB}) \subseteq \mathrm{supp}(\rho_A)$, the joint von Neumann entropy has some interesting properties which are not found on its classical version.

Theorem 28. *Consider a bipartite pure state $|\phi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$, the von Neumann entropy of its marginals are the same:*

$$S(\rho_A) = S(\rho_B), \tag{2.183}$$

where $\rho_A = Tr_B(|\phi\rangle\langle\phi|_{AB})$.

Proof. Any bipartite state $|\phi\rangle_{AB}$ can be written as[10]:

$$|\phi\rangle_{AB} = \sum_i c_i |a_i\rangle |b_i\rangle, \tag{2.184}$$

taking the partial trace over the subsystems:

$$\rho_A = \mathrm{Tr}_B(|\phi\rangle\langle\phi|_{AB}) = \sum_i c_i |a_i\rangle\langle a_i|, \tag{2.185}$$

$$\rho_B = \mathrm{Tr}_A(|\phi\rangle\langle\phi|_{AB}) = \sum_i c_i |b_i\rangle\langle b_i|. \tag{2.186}$$

As the eigenvalues of ρ_A and ρ_B are the same, the entropy also will be the same. □

The same holds for multipartite systems, because it is always possible to divide the system in two subsystems. For example, for a tripartite system described by the state $|\psi\rangle_{ABC}$, the following expression will hold:

$$S(\rho_{AB}) = S(\rho_C), \tag{2.187}$$

$$S(\rho_A) = S(\rho_{BC}), \tag{2.188}$$

$$S(\rho_{AC}) = S(\rho_B). \tag{2.189}$$

Proposition 29 (Additivity)**.** *The von Neumann entropy is additive:*

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma), \tag{2.190}$$

where ρ and σ are density matrices.

Proof. Given the definition of the von Neumann entropy:

$$S(\rho \otimes \sigma) = -\mathrm{Tr}[\rho \otimes \sigma \log_2(\rho \otimes \sigma)]. \tag{2.191}$$

[10]This decomposition is named Schmidt decomposition and shall be presented in next chapter.

If the spectral decomposition of the states are:

$$\rho = \sum_k a_k |a_k\rangle\langle a_k| \tag{2.192}$$

$$\sigma = \sum_l b_l |b_l\rangle\langle b_l|, \tag{2.193}$$

then

$$\log_2(\rho\otimes\sigma) = \log_2\left(\sum_k a_k |a_k\rangle\langle a_k| \otimes \sum_l b_l |b_l\rangle\langle b_l|\right) = \log_2(\rho\otimes\mathbb{I}) + \log_2(\mathbb{I}\otimes\sigma). \tag{2.194}$$

Substituting in Eq.2.191:

$$\begin{aligned} S(\rho\otimes\sigma) &= -\mathrm{Tr}[\rho\otimes\sigma(\log_2(\rho\otimes\mathbb{I}) + \log_2(\mathbb{I}\otimes\sigma))] & (2.195)\\ &= -\mathrm{Tr}[\rho\otimes\sigma\log_2(\rho\otimes\mathbb{I})] - \mathrm{Tr}[\rho\otimes\sigma\log_2(\mathbb{I}\otimes\sigma)] & (2.196)\\ &= -\mathrm{Tr}[\rho\log_2(\rho)] - \mathrm{Tr}[\sigma\log_2(\sigma)] & (2.197)\\ &= S(\rho) + S(\sigma). & (2.198)\end{aligned}$$

□

We also can define a quantum analogous to the mutual information for bipartite states.

Definition 30 (Mutual information)**.** *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A\otimes\mathbb{C}_B)$ the quantum mutual information is defined as:*

$$I(A:B)_{\rho_{AB}} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{2.199}$$

The quantum mutual information of ρ_{AB} quantifies the correlations between the states in qubits.It can be interpreted as the number of qubits which one part must send to the other, to destroy the correlation between them. As the amount of correlations in a quantum state must be positive, we can obtain from the mutual information that:

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}). \tag{2.200}$$

From the Proposition 79 we can obtain that the mutual information will be zero when the bipartite state is a product state $\rho_{AB} = \rho_A\otimes\rho_B$. Then it guarantees that product states are not correlated. It is the quantum analogous to the product of probability vectors: $p(x,y) = p_x p(y)$. In Theorem 28 we calculated that the entropy of the reduced state of pure states are the same. Therefore the mutual information of pure states will be equal to two times the entropy of the reduced state:

$$I(A:B)_{\psi_{AB}} = 2S(\rho_A) = 2S(\rho_B), \tag{2.201}$$

where $\psi_{AB} = |\psi\rangle\langle\psi|_{AB}$ is pure state, and by definition: $S(|\psi\rangle\langle\psi|_{AB}) = 0$.

The quantum analogue to relative entropy defines a measure of distinguishability between quantum states.

Definition 31 (Quantum relative entropy)**.** *Given two density matrices $\rho,\sigma \in \mathcal{D}(\mathbb{C}^N)$, the distinguishability between them can be quantified in qbits via the quantum relative entropy:*

$$S(\rho||\sigma) = Tr[\rho\log_2\rho - \rho\log_2\sigma]. \tag{2.202}$$

It will be zero if $\rho = \sigma$.

In the same way of the classical case, the quantum relative entropy, or von Neumann relative entropy is not symmetric under exchange between the states. The quantum relative entropy is a positive function and it will be real if the $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$, otherwise it diverges to infinity. As we calculated for classical probability distributions, the quantum mutual information also can be written as the a quantum relative entropy.

Proposition 32. *Consider a bipartite state ρ_{AB}, the following expression holds:*

$$I(A:B)_{\rho_{AB}} = S(\rho_{AB}||\rho_A \otimes \rho_B), \tag{2.203}$$

where ρ_A and ρ_B are the reduced states of ρ_{AB}.

Proof. From the definition of the relative entropy we have:

$$S(\rho_{AB}||\rho_A \otimes \rho_B) = -S(\rho_{AB}) - \mathrm{Tr}\{\rho_{AB}\log_2[\rho_A \otimes \rho_B]\}. \tag{2.204}$$

Using the same trick performed in Eq.2.194, we obtain:

$$\begin{aligned} S(\rho_{AB}||\rho_A \otimes \rho_B) &= -S(\rho_{AB}) - \mathrm{Tr}\{\rho_{AB}\log_2[\rho_A \otimes \rho_B]\} & (2.205)\\ &= -S(\rho_{AB}) - \mathrm{Tr}\{\rho_{AB}\log_2[\rho_A \otimes \mathbb{I}_B]\} - \mathrm{Tr}\{\rho_{AB}\log_2[\mathbb{I}_A \otimes \rho_B]\} & (2.206)\\ &= -S(\rho_{AB}) - \mathrm{Tr}\{\rho_A\log_2[\rho_A]\} - \mathrm{Tr}\{\rho_B\log_2[\rho_B]\} & (2.207)\\ &= S(\rho_A) + S(\rho_B) - S(\rho_{AB}). & (2.208) \end{aligned}$$

□

We also can use properties of the mutual information to obtain properties of the von Neumann entropy. A useful property is the concavity of the von Neumann entropy. Before to prove it we first should obtain the expression of the mutual information for classical-quantum states[11].

Proposition 33. *Consider a bipartite state in the form $\rho_{AB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, the von Neumann entropy will be:*

$$S(\sum_x p_x |x\rangle\langle x| \otimes \rho_x) = H(X) + \sum_x p_x S(\rho_x). \tag{2.209}$$

Proof. Given the definition of the von Neumann entropy:

$$S(\sum_x p_x |x\rangle\langle x| \otimes \rho_x) = -\mathrm{Tr}\{\sum_x p_x |x\rangle\langle x| \otimes \rho_x \log_2(\sum_x p_x |x\rangle\langle x| \otimes \rho_x)\}. \tag{2.210}$$

Writing the spectral decomposition of each $\rho_i = \sum_\alpha \lambda_\alpha^i |\lambda_\alpha^i\rangle\langle\lambda_\alpha^i|$, we will have:

$$\begin{aligned} S(\sum_x p_x |x\rangle\langle x| \otimes \rho_x) &= -\mathrm{Tr}\left\{\sum_x p_x |x\rangle\langle x| \otimes \rho_x \log_2\left(\sum_x p_x |x\rangle\langle x| \otimes \sum_\alpha \lambda_\alpha^i \left|\lambda_\alpha^i\right\rangle\left\langle\lambda_\alpha^i\right|\right)\right\} \\ &= -\sum_\alpha \mathrm{Tr}\left\{\sum_x p_x |x\rangle\langle x| \otimes \rho_x |\lambda_\alpha^x\rangle\langle\lambda_\alpha^x| \log_2 p_x\lambda_\alpha^x\right\} & (2.211)\\ &= -\sum_\alpha\sum_x p_x\lambda_\alpha^x \log_2 p_x\lambda_\alpha^x & (2.212)\\ &= -\sum_x\sum_\alpha \lambda_\alpha^x \sum_x p_x \log_2 p_x - \sum_x p_x \sum_\alpha \lambda_\alpha^i \log_2 \lambda_\alpha^i, & (2.213) \end{aligned}$$

as $\sum_\alpha \lambda_\alpha^i = 1$ and $S(\rho_i) = -\sum_\alpha \lambda_\alpha^i \log_2 \lambda_\alpha^i$ thus:

$$S(\sum_x p_x |x\rangle\langle x| \otimes \rho_x) = H(X) + \sum_x p_x S(\rho_x), \tag{2.214}$$

where $H(X) = -\sum_x p_x \log_2 p_x$. □

[11]This class of states shall be discussed with details in next chapter, when we introduce the concept of quantum and classical correlated states.

Proposition 34. *The von Neumann entropy of a state written in a convex combination $\rho = \sum_i p_i \rho_i$ is concave:*

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i). \tag{2.215}$$

Proof. This proof can be obtained calculating the mutual information of a bipartite state in the form:

$$\rho_{XB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i, \tag{2.216}$$

and the mutual information will be:

$$I(X:B)_{\rho_{AB}} = S(\rho_X) + S(\rho_B) - S(\rho_{XB}). \tag{2.217}$$

As $S(\rho_X) = H(X)$ is the Shannon entropy of the random variable X, and using the proposition 33:

$$\begin{aligned} I(X:B)_{\rho_{AB}} &= H(X) + S(\rho_B) - H(X) - \sum_i p_i S(\rho_i) && (2.218) \\ &= S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i). && (2.219) \end{aligned}$$

As the mutual information is positive:

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i). \tag{2.220}$$

□

This property means that the von Neumann entropy cannot decrease under mixing operations.

In contrast with the von Neumman entropy, the relelative entropy always decreases under the action of any quantum channel. This property has an operational meaning: two states are always less distinguishable under the action of noise.

Theorem 35. *Given two density matrices $\rho, \sigma \in \mathcal{D}(\mathbb{C}_A)$ and a quantum channel $\Gamma \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_B)$, the following inequality holds:*

$$S(\rho||\sigma) \geq S(\Gamma(\rho)||\Gamma(\sigma)) \tag{2.221}$$

Proof. As the relative entropy only depends of the spectral decomposition of the states, it is also invariant under the action of isometries operations. Considering an isometry $V \in \mathcal{D}(\mathbb{C}_A, \mathbb{C}_{BE})$, then:

$$S(\rho||\sigma) = S(V\rho V^\dagger||V\sigma V^\dagger). \tag{2.222}$$

The action of a quantum channel can be written, by Stinespring representation, as an isometric operator, where the channel is obtained from the partial trace on the extended space:

$$S(\Gamma(\rho)||\Gamma(\sigma)) = S(\mathrm{Tr}_E(V\rho V^\dagger)||\mathrm{Tr}_E(V\sigma V^\dagger)). \tag{2.223}$$

As the relative entropy is monotonic decreasing under the partial trace[12]

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A), \tag{2.224}$$

hence:

$$S(V\rho V^\dagger||V\sigma V^\dagger) \geq S(\mathrm{Tr}_E(V\rho V^\dagger)||\mathrm{Tr}_E(V\sigma V^\dagger)), \tag{2.225}$$

or by Eq.2.223 and Eq.2.222:

$$S(\rho||\sigma) \geq S(\Gamma(\rho)||\Gamma(\sigma)). \tag{2.226}$$

□

[12]The proof of this inequality can be found on Appendix B of the Ref. [162].

This theorem implies in another property of the quantum mutual information: it is monotonic under local maps. As mutual information quantifies correlations, this means that the amount of correlations reduce under local noise.

Corolary 36. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and quantum channel $\Phi_B \in \mathcal{C}(\mathbb{C}_B, \mathbb{C}_B)$, the mutual information satisfies:*

$$I(A:B)_{\rho_{AB}} \geq I(A:B')_{\mathbb{I}\otimes\Phi(\rho_{AB})}. \tag{2.227}$$

Proof. Given the mutual information:

$$I(A:B)_{\rho_{AB}} = S(\rho_{AB}||\rho_A \otimes \rho_B) \tag{2.228}$$

using the theorem above:

$$I(A:B)_{\rho_{AB}} \geq S(\mathbb{I}_A \otimes \Phi_B(\rho_{AB})||\rho_A \otimes \Phi_B(\rho_B)) = I(A:B')_{\mathbb{I}\otimes\Phi(\rho_{AB})}. \tag{2.229}$$

□

As the states can be written as convex combinations of others states, it is interesting to study the convexity of the information functions. A property that can be used later is the joint convexity of the relative entropy.

Proposition 37. *Consider two states $\rho = \sum_x p_x \rho_x$ and $\sigma = \sum_x p_x \sigma_x$, the relative entropy is joint convex:*

$$S(\rho||\sigma) \leq \sum_x p_x S(\rho_x||\sigma_x). \tag{2.230}$$

Proof. Defining two bipartite states:

$$\rho_{XB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i, \tag{2.231}$$

$$\sigma_{XB} = \sum_i p_i |i\rangle\langle i| \otimes \sigma_i, \tag{2.232}$$

where $\rho_{XB}, \sigma_{XB} \in \mathcal{D}(\mathbb{C}_X \otimes \mathbb{C}_B)$, the reduced density matrices of these states are ρ and σ written above. The relative entropy of these states will be:

$$\begin{aligned}
S(\rho_{XB}|\sigma_{XB}) &= -S(\rho_{XB}) - \mathrm{Tr}\{\sum_i p_i |i\rangle\langle i| \otimes \rho_i \log_2 \sum_j p_j |j\rangle\langle j| \otimes \sigma_j\} && (2.233)\\
&= -S(\rho_{XB}) - \sum_i \mathrm{Tr}\{p_i |i\rangle\langle i| \otimes \rho_i \log_2 p_i |i\rangle\langle i| \otimes \sigma_i\} && (2.234)\\
&= -S(\rho_{XB}) - \sum_i \mathrm{Tr}\{p_i\rho_i \log_2 p_i\sigma_i\} && (2.235)\\
&= -S(\rho_{XB}) - \sum_i \mathrm{Tr}\{p_i\rho_i \log_2 \sigma_i\} - \sum_i \mathrm{Tr}\{p_i\rho_i \log_2 p_i\} && (2.236)\\
&= -H(X) - \sum_i p_i S(\rho_i) - \sum_i p_i \mathrm{Tr}\{\rho_i \log_2 \sigma_i\} - \sum_i \mathrm{Tr}\{p_i \log_2 p_i\} && (2.237)\\
&= \sum_i p_i(-S(\rho_i) - \mathrm{Tr}\{\rho_i \log_2 \sigma_i\}) = \sum_i p_i S(\rho_i||\sigma_i). && (2.238)
\end{aligned}$$

As $S(\rho_{XB}|\sigma_{XB}) \geq S(\rho_B|\sigma_B)$ then:

$$S(\rho||\sigma) \leq \sum_x p_x S(\rho_x||\sigma_x). \tag{2.239}$$

□

This property will be used later to prove that the information that can be accessed from an ensemble, via a given POVM, is convex on the POVM. This is interesting because it proves that the optimal accessible information can always be composed by rank one operators.

In analogy with the classical conditional entropy, preparing a bipartite system ρ_{AB}, we can define a measure for the information which we cannot access of A given B, .

Definition 38. *Consider a bipartite system ρ_{AB}, the quantum conditional entropy is defined as the function:*

$$S(A|B)_{\rho_{AB}} = S(\rho_{AB}) - S(\rho_B). \tag{2.240}$$

One interesting property of the quantum conditional entropy is that it can be non positive. For example, if we consider a bipartite pure state $|\phi\rangle_{AB} = (|00\rangle + |11\rangle)/\sqrt{2}$, the von Neumann entropy of the pure state is zero: $S(|\phi\rangle\langle\phi|_{AB}) = 0$. Nonetheless the reduced state is the maximally mixed state: $\rho_B = \mathbb{I}/2$, whose von Neumann entropy is $S(\mathbb{I}/2) = 1$. Therefore the conditional entropy of this state is negative $S(A|B)_{|\phi\rangle\langle\phi|_{AB})} = -1$. It is out of the scope of the thesis discuss about the negativity of the conditional entropy, although it is very useful and relevant in the context of quantum information. The negative value of the quantum conditional entropy is defined as the *coherent information*:

$$I(A\rangle B) = -S(A|B). \tag{2.241}$$

As the conditional entropy is not symmetric the coherent information is not symmetric too. The conditional entropy has a operational meaning in the state merging protocol, where a tripartite pure state is shared by two experimentalists, one will send part of its state through a quantum channel to the other. The coherent information quantifies the amount of entanglement required to the sender be able to perform the protocol. If it is positive, they cannot use entanglement to perform the state merging, and in the end of it they gain an amount of entanglement [32, 83, 106]. The coherent information also quantifies the capacity of a quantum channel, when optimized over all input states ρ_A, the output state will be ρ_B, this result is known as *LSD theorem* ([52, 99, 145], apud [162]).

The final measure of information presented in this section is named conditional mutual information. Classically it refers to the amount of information of two random variables X and Y have in common, given that a third random variable Z is known. In the quantum case, the conditional mutual information of a tripartite state ρ_{ABC} measures the correlations between A and B conditioned to C.

Definition 39. *Consider a tripartite state $\rho_{ABE} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_C)$, the conditional mutual information is defined by the function:*

$$I(A:B|C)_{\rho_{ABC}} = S(A|C) + S(B|C) - S(AB|C). \tag{2.242}$$

We shall see in next chapter that the quantum discord can be written as a conditional mutual information, whose one part of the state is a classical register.

The quantum conditional mutual information is related to the mutual information via the so called *chain rule*:

$$I(AB:C) = I(B:C|A) + I(A:C), \tag{2.243}$$

for a given tripartite state ρ_{ABC}.

2.7 Quantum states discrimination

In this section we shall discuss the discrimination of quantum states. Consider for example a set of orthogonal states. In this case the states can be perfectly distinguished applying a projective measurement. However in general it is not so simple. We start the discussion with the discrimination between states which compose an ensemble, in other words, for a set of states with a well defined probability to draw each one randomly. After we discuss the discrimination between two states, we shall calculate that the capacity to discriminate the states is related to the trace distance between them.

Suppose two experimentalists Alice and Bob. Alice codify an alphabet in a finite ensemble of quantum systems $\xi = \{p_k, \rho_k\}_{k=0}^{N}$, she will send the alphabet to Bob. Bob must use his best strategy to recover the alphabet. Suppose both of them know the probabilities p_k and the states ρ_k. Therefore the problem which Bob has in his hands is to distinguish between the states, in other words, when he receives the state ρ_x, he should be able to discriminate this state of the others. However the only thing that Bob can do to discriminate the states is performing measurements on them. Bob cannot tomograph ξ, because the state $\rho = \sum_x p_x \rho_x$ has an infinity number of decompositions. Bob also cannot to tomography the states ρ_k because he shall receive each one with probability p_k. Actually Bob should choose a POVM $\mathcal{M}$, with elements $\{M_l\}_{l=0}^{N}$, such that with high probability he can associate an outcome M_x with an input state ρ_x. The probability of success of Bob distinguish the states can be determined by the optimization over all POVMs.

Definition 40. *Given a finite ensemble of quantum states $\xi = \{p_k, \rho_k\}$, $\{\rho_k\} \in \mathcal{D}(\mathbb{C}^N)$, the probability of success to distinguish the states in the ensemble is defined as:*

$$P_{suc}(\xi) = \max_{\mathcal{M}\in\mathcal{P}} \sum_x p_x Tr[M_x \rho_x], \tag{2.244}$$

where $\{M_x\}_x$ are the POVM elements of the measurement $\mathcal{M} \in \mathcal{P}(\mathbb{C}^N)$.

As the probability of success plus the probability of error must be equal to one, we also can write an optimization problem for the probability of error in distinguishing the states in the ensemble as:

$$P_{err}(\xi) = 1 - P_{suc}(\xi) = \min_{\mathcal{M}\in\mathcal{P}} \sum_x \sum_{y\neq x} p_x \mathrm{Tr}[M_y \rho_x]. \tag{2.245}$$

This equation is named *min-error problem*, which is a well studied technique to the discrimination of quantum states [11, 20, 34]. The min-error optimization can be calculated analiticaly for some specific class of states, as we shall see in Chapter 5.

Consider an ensemble $\xi = \{p_k, M_k\}_{k=1}^{M}$, where $\rho_k \in \mathcal{D}(\mathbb{C}_X)$ and $M_k \in \mathcal{P}(\mathbb{C}_\Gamma)$ for every $k = \{0, ..., M\}$. The optimization problem in Eq.2.244 can be written as a semidefinite program:

$$\begin{array}{rc} \text{maximize} & \sum_x p_x \mathrm{Tr}[\rho_x M_X] \\ \text{suject to} & \sum_x M_x = \mathbb{I} \\ & M_x \in \mathcal{P}(\mathbb{C}_\Gamma) \end{array} \tag{2.246}$$

According to Eq.2.152:

$$\begin{array}{rcrc} & \text{Primal Problem} & & \text{Dual Problem} \\ \text{maximize :} & \mathrm{Tr}(AX) & \text{minimize :} & \mathrm{Tr}(BY) \\ \text{Subject to:} & \Phi(X) = B & \text{Subject to:} & \Phi^\dagger(Y) \geq A \\ & X \in \mathcal{P}(\mathbb{C}_A) & & Y \in \mathcal{P}(\mathbb{C}_B) \end{array} \tag{2.247}$$

We can immediately identify:

$$\Phi(X) = \sum_x M_x, \tag{2.248}$$

$$B = \mathbb{I}_\Gamma. \tag{2.249}$$

To obtain $\sum_x p_x \mathrm{Tr}[\rho_x M_X] = \mathrm{Tr}[XA]$ we can write the operators $A \in \mathcal{D}(\mathbb{C}_\Gamma \otimes \mathbb{C}_\Sigma)$ and $X \in \mathcal{P}(\mathbb{C}_\Gamma \otimes \mathbb{C}_\Sigma)$ in the following way:

$$X = \sum_x M_x \otimes |x\rangle\langle x| \tag{2.250}$$

$$A = \sum_y p_y \rho_y \otimes |y\rangle\langle y|, \tag{2.251}$$

therefore $\dim(\mathbb{C}_\Sigma) = M$. This choice implies that the map $\Phi = \mathrm{Tr}_\Sigma$ and $\Phi^\dagger = \otimes \mathbb{I}_\Sigma$. Thus we can rewrite the optimization of Eq.2.246 as the SDP defined by the triple $(\mathrm{Tr}_\Sigma, A, \mathbb{I}_\Gamma)$:

$$\begin{array}{cccc} & \text{Primal Problem} & & \text{Dual Problem} \\ \text{maximize :} & \mathrm{Tr}(AX) & \text{minimize :} & \mathrm{Tr}(Y) \\ \text{Subject to:} & \mathrm{Tr}_\Sigma(X) = \mathbb{I}_\Gamma & \text{Subject to:} & Y \otimes \mathbb{I}_\Sigma \geq A \\ & X \in \mathcal{P}(\mathbb{C}_\Gamma) & & Y \in \mathcal{P}(\mathbb{C}_B) \end{array} \tag{2.252}$$

Actually this SDP satisfies the Slater conditions, once that $X > 0$ and there exist feasible solutions for the dual problem such that $Y \otimes \mathbb{I}_\Sigma > A$. For example, one solution is $Y = \gamma \mathbb{I}_\Gamma$, which is a strictly feasible solution for any $\gamma > \|A\|_\infty$. Therefore there exists a value of γ such that the primal and dual solutions are equal. This means that just there exist only one solution for the min-error optimization problem, and this solution is described by the SDP above.

It is not intuitive to find an analytic solution for this problem, although it is generally easy to check if a given POVM is optimal. A closed expression for the optimal POVM can be obtained from the SDP above. This result is known as *Holevo criterion for measurement optimality*, which we enunciate in the next theorem:

Theorem 41. *Given a POVM measurement $\mathcal{M} \in \mathcal{P}(\mathbb{C}_\Gamma, \mathbb{C}_{\Gamma'})$ with elements $\{M_l\}_l$, it is an optimal measurement in Eq.2.244 if and only if the following expression holds:*

$$\sum_k p_k M_k \rho_k \geq p_l \rho_l, \tag{2.253}$$

for every $l \in \{1, ..., dim(\mathbb{C}_{\Gamma'})\}$.

Proof. The proof of this theorem makes use of the SDP 2.252. Therefore as the solution is strictly feasible we shall prove the reverse and converse solution proving that the optimality of the primal problem implies in the statement in Eq.2.253 and optimality of the dual problem also implies in the statement. If $\mathcal{M}$ is an optimal solution the following operator is the optimal primal solution of the problem 2.252:

$$X = \sum_k M_k \otimes |k\rangle\langle k| \,. \tag{2.254}$$

Using the complementary slackness:

$$(Y \otimes \mathbb{I}_\Sigma)X = AX. \tag{2.255}$$

Taking the trace over Σ:

$$\mathrm{Tr}((Y \otimes \mathbb{I}_\Sigma)X) = Y\mathrm{Tr}_\Sigma(X) = Y \tag{2.256}$$

and

$$\mathrm{Tr}_\Sigma(AX) = \sum_x p_x M_x \rho_x. \tag{2.257}$$

Therefore given an optimal solution for the primal problem this implies in:

$$Y = \sum_x p_x M_x \rho_x, \tag{2.258}$$

which from the dual problem of the optimization problem 2.252 is $Y \geq p_l \rho_l$ for every l.

The reverse can be proved if we assume that $Y = \sum_x p_x M_x \rho_x$ is the optimal solution for the dual problem. If the Slater's condition are satisfied, by strong duality, the primal solution and dual solution are the same. Therefore:

$$\mathrm{Tr}(XA) = \mathrm{Tr}(Y) = \mathrm{Tr}(\sum_x p_x M_x \rho_x), \tag{2.259}$$

which implies that $X = \sum_k M_k \otimes |k\rangle\langle k|$ is the optimal solution for the primal problem. □

This theorem implies in another property of the POVM for the min-error problem, which often in the literature appears as a condition for the optimality of the POVM.

Corolary 42. *If the POVM elements $\{M_l\}_l$ satisfy the condition in Eq.2.253, for an ensemble $\xi = \{p_k, M_k\}_k$, these also must satisfy:*

$$M_k(p_k\rho_k - p_l\rho_l)M_l = 0. \tag{2.260}$$

Proof. If Eq.2.253 is valid the following also is:

$$\sum_k p_k M_k \rho_k - p_l\rho_l \geq 0. \tag{2.261}$$

Multiplying M_l on the right and summing over l:

$$\sum_l (\sum_k p_k \mathrm{Tr}[M_k\rho_k] - p_l\rho_l)M_l = \sum_{l,k}(\sum_k p_k M_k\rho_k M_l - p_l M_k \rho_l M_l) \tag{2.262}$$

$$= \sum_{l,k}\sum_k p_k M_k \rho_k M_l - \sum_{l,k} p_l M_k \rho_l M_l \tag{2.263}$$

$$= 0, \tag{2.264}$$

hence:

$$M_k(p_k\rho_k - p_l\rho_l)M_l = 0. \tag{2.265}$$

□

Even though it is not possible to use the Holevo criterion for measurement optimality to obtain an expression for the optimal POVM in general, it is possible to use it to check if the solution is the optimal via necessary and sufficient conditions. This will be useful in Chap.5, where we shall use a measurement named pretty good measurement, which is not optimal although it is pretty good, as the very name says, to check the optimality of it.

2.7.1 Discriminating a pairs of states

Suppose two experimentalists Alice and Bob. Alice will send a binary alphabet $\{0, 1\}$ to Bob, this alphabet is described by the random variable Y. Alice prepares the random variable such that she draws the binary states as:

- She draws 0 with probability λ;
- She draws 1 with probability $1 - \lambda$.

To send the binary information to Bob, Alice must codify the information in a physical system. Alice chooses to codify the information in a quantum system described by two quantum states $\rho_0, \rho_1 \in \mathcal{D}(\mathbb{C}^N)$ of a random variable X. Then if Alice draws the value $Y = 0$, she prepares X in the state ρ_0 and sends it to Bob; and if she draws the values $Y = 1$, she prepares X in the state ρ_1 and send it to Bob. Now Bob must correctly determine the bit stored in Y performing a measurement in the quantum state received from Alice. To Bob learn about Y just measuring X, he must be able to distinguish between the states (way suppose λ, ρ_0 and ρ_1 are known). Because he must know what state was sent by Alice. Suppose Bob chooses a POVM $\mathcal{M}$ composed by two elements M_0, M_1. Bob hopes that when he measures M_x the state sent by Alice is ρ_x, therefore the probability of success which represents the hope of Bob to distinguish the state with the measurement $\mathcal{M}$, is the function:

$$P_{suc}^{\mathcal{M}} = p_0 p(0|0) + p_1 p(1|1), \tag{2.266}$$

where $p_0 = \lambda$ and $p_1 = 1 - \lambda$ are the probabilities that Bob receive the state ρ_0 and ρ_1 respectively, and $p(x|x) = \mathrm{Tr}(M_x\rho_x)$ is the probability that Bob measures M_x when the input is ρ_x.

The main objective here will be prove that the probability of success that Bob correctly performs the discrimination depends on the trace distance of ρ_0 and ρ_1. First of all we shall prove the following lemma:

Lemma 43. *Given a set of positive operators* $\{X_i\}_{i=1}^{M} \subset \mathcal{P}(\mathbb{C}^N)$ *and a vector* $\vec{v} \in \mathbb{C}^M$ *with components* $\{v_i\}_{i=1}^{M}$*, the following relation is valid:*

$$\| \sum_i v_i X_i \|_\infty \leq \|\vec{v}\|_\infty \| \sum_i X_i \|_\infty. \tag{2.267}$$

Proof. Given an isometry $A \in \mathcal{U}(\mathbb{C}^N, \mathbb{C}^N \otimes \mathbb{C}^E)$:

$$A = \sum_i \sqrt{X_i} \otimes |i\rangle\,, \tag{2.268}$$

therefore:

$$\sum_i v_i X_i = \sum_i v_i A^\dagger (\mathbb{I} \otimes |i\rangle\langle i|) A. \tag{2.269}$$

Taking the infinity norm of this operator:

$$\begin{aligned} \| \sum_i v_i X_i \|_\infty &= \| \sum_i v_i A^\dagger (\mathbb{I} \otimes |i\rangle\langle i|) A \|_\infty && (2.270)\\ &\leq \|A^\dagger\|_\infty \| \sum_i v_i \,|i\rangle\langle i|\, \|_\infty \|A\|_\infty && (2.271)\\ &= \|A^\dagger\|_\infty \|A\|_\infty \|\vec{v}\|_\infty. && (2.272) \end{aligned}$$

Using the property of the infinity norm $\|A^\dagger\|_\infty \|A\|_\infty = \|A^\dagger A\|_\infty$ and

$$\|A^\dagger A\|_\infty = \|(\sum_i \sqrt{X_i} \otimes \langle i|)(\sum_j \sqrt{X_j} \otimes |j\rangle)\|_\infty = \| \sum_i X_i \|_\infty. \tag{2.273}$$

Substituting Eq.2.273 in Eq.2.272 we prove the lemma. □

The interplay between the scenario described above and the trace distance is given by the Holevo-Helstron inequality.

Theorem 44 (Holevo-Helstron)**.** *Given two states* $\rho_0, \rho_1 \in \mathcal{D}(\mathbb{C}^N)$ *where each one can be draw with the probability* λ *and* $1-\lambda$ *respectively, and a POVM* $\mathcal{M}$ *with elements* $\{M_0, M_1\} \in \mathcal{P}(\mathbb{C}^N)$*, the following expression holds:*

$$\lambda Tr(M_0\rho_0) + (1-\lambda) Tr(M_1\rho_1) \leq \frac{1}{2} - \frac{1}{2}\|\rho_0 - \rho_1\|_1, \tag{2.274}$$

where $\|\rho_0 - \rho_1\|_1$ *is the trace distance between the states.*

Proof. First we define two new operators:

$$\begin{aligned} X &= \lambda\rho_0 - (1-\lambda)\rho_1, && (2.275)\\ \rho &= \lambda\rho_0 + (1-\lambda)\rho_1. && (2.276) \end{aligned}$$

Rewriting the left hand side of Eq.2.274 as:

$$\lambda \mathrm{Tr}(M_0\rho_0) + (1-\lambda)\mathrm{Tr}(M_1\rho_1) = \frac{1}{2}\mathrm{Tr}[(M_0 + M_1)\rho] + \frac{1}{2}\mathrm{Tr}[(M_0 - M_1)X]. \tag{2.277}$$

Using the Holder inequality in both terms on the right hand side:

$$\frac{1}{2}\mathrm{Tr}[(M_0+M_1)\rho] = \frac{1}{2} \quad (2.278)$$
$$\frac{1}{2}\mathrm{Tr}[(M_0-M_1)X] \leq \frac{1}{2}\|M_0-M_1\|_\infty\|X\|_1. \quad (2.279)$$
$$(2.280)$$

As $M_0 - M_1 = \vec{v}\cdot\vec{M}$ for $\vec{v} = (1,-1)$ and $\vec{M} = (M_0, M_1)$, therefore using Lemma 43:

$$\|M_0-M_1\|_\infty \leq \|\vec{v}\|_\infty\|M_0+M_1\|_\infty = 1. \quad (2.281)$$

Hence the expression 2.277 satisfies:

$$\lambda\mathrm{Tr}(M_0\rho_0) + (1-\lambda)\mathrm{Tr}(M_1\rho_1) \leq \frac{1}{2} + \frac{1}{2}\|X\|_1 = \frac{1}{2} + \frac{1}{2}\|\rho_0-\rho_1\|_1. \quad (2.282)$$

To complete the proof, we should prove that there is a measurement such that the equality is attained. We can calculate it using the same arguments in the proof of Proposition 13. There exists a projective measurement: $\{\Pi_+, \Pi_-\}$ such that:

$$\Pi_+(\rho_0-\rho_1)\Pi_+ = \alpha_+ \quad (2.283)$$
$$\Pi_-(\rho_0-\rho_1)\Pi_- = \alpha_-, \quad (2.284)$$

where α_+ and α_- are matrices whose eigenvalues are the absolute values of the positive and negative eigenvalues of $X = \rho_0 - \rho_1$ respectively. Thus Eq.2.277 resulting:

$$\frac{1}{2} + \frac{1}{2}\mathrm{Tr}[(\Pi_+ - \Pi_-)X] = \frac{1}{2}(\alpha_+ + \alpha_-). \quad (2.285)$$

As the spaces which α_+ and α_- spam are orthogonal:

$$\|X\|_1 = \mathrm{Tr}(|X|) = \mathrm{Tr}|\alpha_+ - \alpha_-| = \mathrm{Tr}(\alpha_+) + \mathrm{Tr}(\alpha_-), \quad (2.286)$$

that implies:

$$\frac{1}{2}\mathrm{Tr}[(\Pi_+ - \Pi_-)X] = \frac{1}{2}\|X\|_1, \quad (2.287)$$

and proves the theorem. □

The best strategy to distinguish the states is:

$$p_{suc} = \max_{\substack{\{M_0, M_1\}\in\mathcal{P} \\ M_0+M_1=\mathbb{I}}} \left[\mathrm{Tr}(M_0\rho_0) + \mathrm{Tr}(M_1\rho_1)\right], \quad (2.288)$$

As we have seen in the theorem, there exist a POVM such that the equality is attained, therefore:

$$p_{suc} = \frac{1}{2} + \frac{1}{2}\|\lambda\rho_0 - (1-\lambda)\rho_1\|. \quad (2.289)$$

The best case for Bob is when the states are orthogonal and the $\lambda = 1/2$. For this case the probability of success is equal to one and Bob can always distinguish the states, and consequently decode the binary message sent by Alice. The worst case is when the trace distance between the states is zero, for this case the probability of success is a half, which means that the best thing to do is to choose randomly between any of the the results, and associate it to any state.

2.8 Accessible information

As we are discussing information, we must be able to quantify in bits the amount of information which one can extract from an ensemble of states. It is a way to quantify the degree of discriminality between the states.

Suppose two experimentalists Alice and Bob, where the former will send a message for the latter. She encodes the message in a set of quantum states given by the quantum ensemble $\xi = \{p_i, \rho_i\}_{i=0}^{M-1}$, where p_i is the probability to obtain the state $\rho_i \in \mathcal{D}(\mathbb{C}_B)$. The ensemble can be represented as a classical-quantum bipartite state $\rho_{XB} \in \mathcal{D}(\mathbb{C}_X \otimes \mathbb{C}_B)$ as: $\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, where $\{|x\rangle\}_{x=0}^{M-1}$ is an orthonormal basis in $\mathbb{C}_X$. Alice gets the register $\{x\}_{x=0}^{M-1}$ and sends the message ξ to Bob, he must perform a measurement $\mathcal{M} \in P(\mathbb{C}_B, \mathbb{C}_Y)$, where $\dim \mathbb{C}_Y = M$, to extract the information codified in ξ. The amount of information which can be extracted by Bob, performing the measurement $\mathcal{M}$, is given by the correlations between Alice and Bob after the action of the measurement map.

Definition 45. *Given the measurement map $\mathbb{I}_X \otimes \mathcal{M}_B(\rho_{XB}) = \sum_x p_x |x\rangle\langle x| \otimes \mathcal{M}(\rho_x)$, where $\mathcal{M}(\rho_x) = \sum_{y=0}^{M-1} Tr[\rho_x M_y] |y\rangle\langle y|$. The amount of information $I(\xi : \mathcal{M})$ extracted from the ensemble ξ, performing the measurement $\mathcal{M}$, is defined as the mutual information of the state $\rho_{XY} = \mathbb{I}_X \otimes \mathcal{M}_B(\rho_{XB})$:*

$$I(\xi : \mathcal{M}) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}), \tag{2.290}$$

where $S(\rho)$ is the von Neumann entropy of the state ρ.

As the post-measurement state will be a probability vector given by the joint probability $p(x,y) = p_x p(y|x) = p_x Tr[\rho_x M_y]$, where the conditional probability to obtain the result y given the state x is given by $p(y|x) = Tr[\rho_x M_y]$, therefore the information $I(\xi : \mathcal{M})$ extracted from the ensemble ξ performing the measurement $\mathcal{M} = \{M_y\}_{y=0,\dots,M-1}$ must be a classical Shannon mutual information for joint probability $\{p(x,y)\}_{x,y=0,\dots M-1}$:

Proposition 46. *The mutual information of the post-measurement state is given by the Shannon mutual information of the probability distribution given by the vector $\{p(x,y)\}_{x,y=0,\dots,M-1}$:*

$$I(\xi : \mathcal{M}) = I(X:Y) = H(X:Y) = \sum_{x,y} p(x,y) \log[\frac{p(y|x)}{\sum_z p(z)p(y|z)}]. \tag{2.291}$$

Proof. The mutual information of the post-measurement state $\rho_{XY} = \sum_{xy} p(x,y) |x\rangle\langle x| \otimes |y\rangle\langle y|$ is given by:

$$I(X:Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}) = -\sum_x p_x \log p_x - \sum_y p(y) \log p(y) + \sum_{x,y} p(x,y) \log p(x,y), \tag{2.292}$$

where $p_x = \sum_y p(x,y)$ and $p(y) = \sum_x p(x,y)$. We can rewrite the last expression as:

$$I(X:Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_x p(y)} \tag{2.293}$$

$$= \sum_{x,y} p(x,y) \log \frac{p_x p(y|x)}{p_x p(y)} \tag{2.294}$$

$$= \sum_{x,y} p(x,y) \log \frac{p(y|x)}{\sum_z p_z p(y|z)}. \tag{2.295}$$

□

The maximum amount of information which can be extracted from an ensemble is named accessible information, by the last proposition, it represents the maximal correlations between the input probability distribution and the output.

Definition 47. *The accessible information $I(\xi)$ for the ensemble ξ is defined as the maximum amount of information which can be extracted from the ensemble optimizing over all rank-1 POVMs:*

$$I(\xi) = \max_{\mathcal{M}=P(\mathbb{C}_B,\mathbb{C}_Y)} I(\xi : \mathcal{M}). \tag{2.296}$$

The analytical optimal Eq.2.296 is not known, for some classes of states it can be calculated analytically as we shall discuss in the next section. Nonetheless there is a necessary, but not sufficient, condition for a given POVM $\mathcal{M} = \{M_y\}_{y=0,\ldots,M-1}$ to maximize Eq.2.296 for an ensemble $\xi = \{p_{,k}, \rho_k\}_{k=0,\ldots,M-1}$, it must satisfies the following conditions [2]:

$$M_i[F_j - F_i]M_j = 0, \tag{2.297}$$

for $F_i = \sum_k p_k \rho_k \log\left[p(i|k)/\sum_l p_l p(j|l)\right]$. This condition is related to the inflection point $\delta I(\xi) = 0$, where δ is the variation with respect to $\{M_y\}_{y=0,\ldots,M-1}$. A sufficient condition equivalent to $\delta^2 I(\xi) \leq 0$ is not known. Therefore given a POVM which satisfies Eq.2.297, it is an extremal point for the accessible information, but we cannot guarantee that it is a minimum or a maximum point. Even numerically this is a hard computational problem, once that the mutual information is not a linear function and the space of POVMs has the square of the dimension of the space of states.

As the space of POVMs is a convex space, thus we can write a POVM $\mathcal{P}$ as a convex combination of other POVMs $\mathcal{P}_k$, by means of direct sum:

$$\mathcal{Q} = \bigoplus_k \lambda_k \mathcal{P}_k, \tag{2.298}$$

for $\sum_k \lambda_k = 1$, which is equivalent to refine the measurement. An example is the post-processing, where each element of the POVM is decomposed in the singular value decomposition originating another POVM which can be viewed as a convex combination of other POVMs restricted to projective measurements [31].

Proposition 48. *The information extracted from a finite ensemble of quantum states $\xi = \{p_k, \rho_k\}$ performing a measurement $\mathcal{M}$ is a convex function in the POVM.*

Proof. This proof comes directly from the convexity of the relative entropy in proposition 37, and from the definition of the extracted information $I(\xi : \mathcal{M})$. □

2.8.1 Holevo's bound

The accessible information quantifies the information which can be extracted from a quantum ensemble. But how much information can be codified in a quantum system? The function which quantifies the maximum amount of information that can be codified in quantum ensemble is measured by the Holevo information.

Definition 49. *Given a finite ensemble of quantum states $\xi = \{p_x, \rho_x\}_x$, the Holevo information is defined as.*

$$\chi(\xi) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x), \tag{2.299}$$

where $\rho_x \in \mathcal{D}(\mathbb{C}^N)$ and $\sum_x p_x = 1$.

As the Holevo's information quantify the maximum amount of bits which can be codified in a quantum ensemble, we hope that the maximum amount of bits which can extracted from the ensemble is at least equal to this quantity. Which feeds our hopefulness is named Holevo's bound and is stated by the following theorem.

Theorem 50 (Holevo's bound). *Given a finite quantum ensemble $\xi = \{p_x, \rho_x\}_x$, the accessible information is upper bounded by the Holevo's information:*

$$I(\xi) \leq \chi(\xi), \tag{2.300}$$

where the quantity $\chi(\xi) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$ *is the Holevo information.*

Proof. Given the state which represents the ensemble ξ:

$$\rho_{XB} = \sum p_x |x\rangle\langle x| \otimes \rho_x^B, \tag{2.301}$$

The mutual information for this state is:

$$I(X:B)_{\rho_{XB}} = S(\rho_X) + S(\rho_B) - S(\rho_{XB}), \tag{2.302}$$

where $S(\rho_X) = H(X)$ and $S(\rho_{XB}) = H(X) + \sum_x p_x S(\rho_x^B)$, then:

$$I(X:B)_{\rho_{XB}} = S(\rho_B) - \sum_x p_x S(\rho_x^B) = \chi(\xi). \tag{2.303}$$

Now we must prove that this quantity is greater than the accessible information. Indeed it comes from the monotonic property of the mutual information. For a given measurement $\mathcal{M}$ with measurements operators $\{M_y\}_y$, the after measured state is

$$\rho_{XY} = \mathbb{I}_X \otimes \mathcal{M}(\rho_{XB}) = \sum_x p_x \mathrm{Tr}(\rho_x M_y) |x\rangle\langle x| \otimes |y\rangle\langle y|. \tag{2.304}$$

The after measured state suffers the action of a quantum channel, named as measurement channel, therefore as the mutual information is monotonic under local maps:

$$I(X:Y)_{\mathbb{I}_X \otimes \mathcal{M}(\rho_{XB})} \leq I(X:B)_{\rho_{XB}} = \chi(\xi). \tag{2.305}$$

As it is valid for any measurement it will remains valid for the measurement which optimizes the accessible information. □

The Holevo's quantity measures the amount of classical information which one can codify in a quantum system, described by a finite ensemble of quantum states. Therefore there is a parallel between the amount of classical information which can be codified in a quantum system, described by the Holevo's quantity, and the amount of classical information which can be extracted from a quantum system, described by the accessible information. When we write an ensemble $\xi = \{p_x, \rho_x\}_x$ as a bipartite stat $\rho_{XB} = \sum p_x |x\rangle\langle x| \otimes \rho_x^B$ we are describing the ensemble as a quantum system correlated with a classical register X.

As the mutual information quantifies correlations, the Holevo's quantity quantifies the correlations between the classical register X and the quantum system, described by the ensemble ξ. On other hand as the accessible information is a classical mutual information it is quantifying the amount of classical correlation between the classical register X and the quantum system B, which means that the measurement process destroyed the quantum correlations between the classical and quantum system. We shall discuss later in the text that the difference between the Holevo's information and the accessible information quantifies the amount of quantum correlations destroyed by the measurement process.

Chapter 3

Quantum correlations

3.1 Quantum Entanglement

The problem of quantum entanglement is related to determine if a quantum state of a composed system is separable or not. In this way, to determine if a state is separable it necessary find if it is inside or outside the set of the separable states. Therefore, the quantum entanglement is not a property of a density matrix by itself, it is a property of the composed space of density matrices. We can understand it if we take an example a positive, semi-definite and trace 1 4×4 matrix. What can we say about entanglement of it? Nothing, we must first define the partitioning of it, what means that: we must define the space of density matrix where it lives. The, if I said that the space of density matrix is partitioned as $2 \otimes 2$ system, hence I should to determine if the state is a separable or not. The degree of inseparability is named *amount of entanglement*, in this section we present some measure of entanglement: negativity, generalized robustness of entanglement, random robustness of entanglement, entanglement formation, entanglement cost, distillable entanglement. The first three of them we show that come from a geometrical approach related to the entanglement witness.

3.1.1 Separable states

Pure states

We consider two systems A and B, often we named the experimentalists responsible by the systems as Alice and Bob respectively. The state of the systems A and B is described by a density matrix on a Hilbert space. In this way we can consider two finite Hilbert spaces $\mathbb{C}_A$ and $\mathbb{C}_B$, and a basis in each one:

$$\{|a_i\rangle\}_{i=0}^{|A|-1} \in \mathbb{C}_A; \tag{3.1}$$

$$\{|b_k\rangle\}_{k=0}^{|B|-1} \in \mathbb{C}_B, \tag{3.2}$$

where $|A| = \dim(\mathbb{C}_A)$ and $|B| = \dim(\mathbb{C}_B)$. The global, system composed by A and B, can be obtained via the tensor product between the basis in the Hilbert space of each system:

$$\{|a_i, b_k\rangle\}_{i,j=0}^{|AB|-1} = \{|a_i\rangle \otimes |b_k\rangle\}_{i,k=0}^{|A|-1,|B|-1}, \tag{3.3}$$

hence the dimension of the composed system is the product of the dimension: $|AB| = \dim(\mathbb{C}_{AB}) = \dim(\mathbb{C}_A) \cdot \dim(\mathbb{C}_B)$. The Hilbert space of the composed system is denoted as $\mathbb{C}_{AB} = \mathbb{C}_A \otimes \mathbb{C}_B$. A pure state which describes the state of the composed system can be decomposed in the basis in Eq.3.3:

$$|\psi\rangle_{AB} = \sum_{i,k} c_{i,k} |a_i\rangle \otimes |b_k\rangle . \tag{3.4}$$

From this expression we can realize that: in general a pure state which describes a composed system cannot be written as the product of the state of each system. In other words, suppose the system A is describe by the state $|\alpha\rangle_A = \sum_i a_i |a_i\rangle \in \mathbb{C}_A$ and $|\beta\rangle_B = \sum_k b_k |b_k\rangle \in \mathbb{C}_B$. The composed system is described by the state:

$$|\alpha\rangle \otimes |\beta\rangle = \sum_{i,k} a_i b_k |a_i\rangle \otimes |b_k\rangle . \tag{3.5}$$

It is the particular case where the coefficients in Eq.3.4 are $c_{i,k} = a_i \cdot b_k$. If a composed system can be written as Eq.3.5 it is called a *product state*, and there is no correlations between A and B. It can be checked easily via the mutual information of the state, which is clearly zero once that the von Neumman entropy of the pure state is zero. In the pure states subspace, the set of product states is a tiny set ([131,132], apud [170]).

If a state of a composed system cannot be written as Eq.3.5 it is named a *entangled state*. For bipartite systems, the entanglement of pure states can be characterized and quantified via the *Schmidt decomposition*.

Theorem 51 (Schmidt decomposition)**.** *For any bipartite pure state $|\psi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$, there exists a basis $\{|a_i, b_i\rangle\}_{i=0}^{r-1} \in \mathbb{C}_A \otimes \mathbb{C}_B$ such that the state can be written as:*

$$|\psi\rangle_{AB} = \sum_{i=0}^{r-1} c_i |a_i, b_i\rangle , \tag{3.6}$$

where the coefficients $\{c_i\}_i$ are real numbers named Schmidt coefficients and $\sum_i c_i^2 = 1$, the number $r \leq \min(|A|, |B|)$ is named Schmidt rank.

Proof. Decomposing the state $|\psi\rangle_{AB}$ in the canonical basis:

$$|\psi\rangle_{AB} = \sum_{k,l} a_{k,l} |k, l\rangle , \tag{3.7}$$

then, we can approach the coefficients $\{a_{k,l}\}_{k,l}$ as the coefficients of $|A| \times |B|$ matrix a with singular value decomposition $a = udv$, therefore the coefficients can be written as:

$$a_{k,l} = \sum_i u_{k,i} d_{i,i} v_{i,l}. \tag{3.8}$$

Substituting this expression in Eq.3.7 we have:

$$|\psi\rangle_{AB} = \sum_i \sum_{k,l} u_{k,i} d_{i,i} v_{i,l} |k, l\rangle , \tag{3.9}$$

relabeling $|a_i\rangle = \sum_k u_{k,i} |k\rangle$, $|b_i\rangle = \sum_l v_{i,l} |l\rangle$ and $c_i = d_{i,i}$, we have the Schmidt decomposition:

$$|\psi\rangle_{AB} = \sum_{i=0}^{r-1} c_i |a_i, b_i\rangle . \tag{3.10}$$

From the singular value decomposition is easy to check that the states $\{|a_i, b_i\rangle\}_i$ form an orthonormal basis, and $\sum_i c_i^2 = 1$ because $\sum_{i,j} a_{i,j}^2 = 1$. □

We can realize from the Schmidt decomposition that the entropy reduced matrices of a bipartite pure states are the same for the subsystems A and B:

$$S(\mathrm{Tr}_A(\psi_{AB})) = S(\mathrm{Tr}_B(\psi_{AB})) = -\sum_i c_i^2 \log_2(c_i^2), \tag{3.11}$$

where $\psi_{AB} = |\psi\rangle\langle\psi|_{AB} = \sum_{i,j} c_i c_j |a_i, b_i\rangle\langle a_j, b_j|$.

Given the Schmidt decomposition of a bipartite pure state $|\psi\rangle_{AB}$, the entanglement of the state can be characterized via the Schmidt rank. It is possible to understand this just looking over the product state in Eq.3.5. For this kind of states we just have only one Schmidt coefficient $c_0 = 1$, the other are equal to zero. Therefore the Schmidt rank is equal to 1 for product states. We also can define a measure of entanglement for bipartite pure state calculated via the Schmidt decomposition, it is named *entanglement entropy*.

Definition 52 (Entanglement entropy)**.** *Given a bipartite state $|\psi\rangle_{AB}$ the amount of entanglement in the state is quantified by the von Neumann entropy of the reduced state:*

$$E(|\psi\rangle) = S(\rho_A) = -\sum_i c_i^2 \log_2(c_i^2). \tag{3.12}$$

The entanglement entropy is zero for product states, and its maximal value is $\log_2 r$. The states which the entanglement entropy is maximum are named the *maximally entangled states*, and their Schmidt decompositions are in the form:

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{r}} \sum_{i=0}^{r-1} |a_i, b_i\rangle \,, \tag{3.13}$$

where r is the Schmidt rank, for the maximally entangled states it is $r = \min(|A|, |B|)$. For bipartite systems where $|A| = |B| = d$ the Schmidt rank is just equal to d, and the maximally entangled state is written as:

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |a_i, b_i\rangle \,. \tag{3.14}$$

Maximally entangled estates are often called *e-bits*. For two qubits systems the maximally entangled states are named *Bell states*, and they form a orthonormal basis in the 4-dimensional Hilbert space.

Mixed states

The concept of product state can be generalized for mixed state. Considering a composed system represented by the state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, it is called a *product state* if can be written as:

$$\rho_{AB} = \rho_A \otimes \rho_B, \tag{3.15}$$

where $\rho_A \in \mathcal{D}(\mathbb{C}_A)$ and $\rho_B \in \mathcal{D}(\mathbb{C}_B)$ are the states of the system A and B respectively. The product state for mixed states is also no correlated, we can check it calculating the mutual information of the state ρ_{AB}, that as we showed in the last chapter is zero for product states.

As the space of quantum states is a convex set, the convex combination of states will also be a quantum state. Then we can generalize the notion of product states taking the convex combination of them. The resulting state is named a *separable state* [160].

Definition 53 (Separable states)**.** *Considering a composed system described by the state $\sigma \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, it is named separable state if and only if can be written as:*

$$\sigma = \sum_{i,j} p_{i,j} \sigma_i^A \otimes \sigma_j^B, \tag{3.16}$$

where $\sigma_i^A \in \mathcal{D}(\mathbb{C}_A)$ and $\sigma_j^B \in \mathcal{D}(\mathbb{C}_B)$.

The quantum channels which let the set of separable state invariant is named *local operations and classical communication* (LOCC). The set of separable states form a subspace in the space of density matrices, will be denote it as $Sep(\mathbb{C}_{AB})$. The space of separable states has nonzero volume, then we can think of it as a $(|A| \cdot |B|)$-dimensional ball around the maximal mixture state [170].

The separable state can be easily extended to multipartite systems. Considering a n-partite system it is named m-separable if it can be decomposed in a convex combination of product states composed by m parties. Where naturally $m \leq n$.

As for bipartite mixed states as for any multipartite state the Schmidt decomposition is not valid, then the entanglement entropy cannot be used. therefore, how can we characterize and quantify the entanglement for any mixed state? We shall present three methods in next sections.

3.1.2 Partial transpose

In last chapter when we discuss the completely positive condition for a given map be a quantum channel we use as example the transposition operation, which is positive but not completely positive. This property of the transposition map can be used to characterize the separability of a quantum state.

Theorem 54 (Peres Criterion)**.** *If a bipartite state $\sigma_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is separable then:*

$$\mathbb{I} \otimes T(\sigma_{AB}) \geq 0 \quad \textit{and} \quad T \otimes \mathbb{I}(\sigma_{AB}) \geq 0, \tag{3.17}$$

where T is the transposition map.

Proof. Considering a bipartite separable state σ_{AB}:

$$\sigma_{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B. \tag{3.18}$$

As the transposition operation is a positive map $T(\rho_k^A) \geq 0$ and $T(\rho_k^B) \geq 0$, then the theorem holds. □

This criterion was introduced by A. Peres [126], and gives a necessary condition for a bipartite state be separable, nonetheless it is not sufficient to guaranty that a given state is separable.

From the Peres criterion it is possible to write a necessary and sufficient condition for $2 \otimes 2$ and $2 \otimes 3$ systems [81].

Theorem 55 (Horodecki criterion)**.** *Considering a Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, or $\mathbb{C}^2 \otimes \mathbb{C}^3$, a given density matrix ρ_{AB}, on any of this spaces, is separable if and only if $\mathbb{I} \otimes T(\rho_{AB}) \geq 0$. Otherwise it is entangles.*

Proof. A pedagogical proof of this theorem can be found in the M. Lewenstein lectures [95]. □

These two criteria are usually attached in only one condition for separability, that is name *Peres-Horodecki criterion*. The Peres-Horodecki criterion makes the calculation of entanglement for two qubits systems be analytic. In other words, it is possible to define analytical measure for the entanglement for two qubits, all of them based on the Peres-Horodecki criterion, for example Concurrence [77], entanglement of formation [163] and negativity [157]. The entanglement of formation is a entropic measure of entanglement which the analogous to the entanglement entropy for mixed states, it is just analytical for two qubits, where it can be written in function of the concurrence. We shall present it in Sec.3.1.4. The concurrence and negativity are measures of entanglement which hold just for systems with dimension $2 \otimes 2$ and $2 \otimes 3$. In this thesis we are particularly interested in the negativity, because it shall be use to compare quantum entanglement and the geometric measure of quantum discord in the next chapter. The negativity is defined as the sum of the negative eigenvalues of the partial transposed matrix [157].

Definition 56. *Considering a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, for $|A| \cdot |B| = 4$ or 6, one defines the Negativity as:*

$$\mathcal{N}(\rho_{AB}) = \frac{1}{2}\left(\|\mathbb{I} \otimes T(\rho_{AB})\|_1 - 1\right), \tag{3.19}$$

where T is the transposition operation.

Why is the Peres criterion necessary but no sufficient to characterize a separable state? Because for systems with dimension different of $2 \otimes 2$ and $2 \otimes 3$ there exist entangled states with positive partial transpose (PPT - entangled states). This states are ofter called *bound entangled states*, because they are not distillable[1]. A recipe to construct a class of PPT-entangled states is via a special orthogonal and incomplete basis named unextendible product basis (UPB) [15,55].

An interesting measure of entanglement is named *Robustness of entanglement*, it quantifies the amount of mixture with a separable state needed to destroy the entanglement of the system [158]. Actually it quantifies the entanglement in a geometrical sense, once that the space of separable states is a convex set. Nonetheless we are not interested in the robustness of entanglement, we are indeed in it general form: the *generalized robustness of entanglement*. This quantifies the amount of mixture with any state needed to destroy the entanglement [149]. Formally it is defined as:

Definition 57 (Generalized robustness of entanglement)**.** *Consider a n-partite state $\rho \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$, and another state ρ_s, we call general robustness of entanglement of ρ the minimal value of $s \geq 0$, such that the state:*

$$\rho(s) = \frac{1}{1+s}(\rho + s\rho_s) \tag{3.20}$$

is separable.

The parameter s shall be zero if the state ρ is separable, and it is always finite, once that the set of separable state is a ball around the identity, as we are interested in the minimal value, the minimal s will be found in border of the separable states [158]. In this way the state ρ_s is the state whose the entanglement of ρ is most sensible. We also can present a particular case of generalized robustness: the *renadom robustness*. It is defined as the minimal mixture of the state with the maximally mixture state.

Definition 58. *We named random robustness of ρ the minimal value of $s \geq 0$ such that the state:*

$$\rho(s) = \frac{1}{1+s}(\rho + s\frac{\mathbb{I}}{N}), \tag{3.21}$$

where $N = dim(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$.

The operational definition of the random robustness is related to the amount of white noise needed to wash out all the entanglement in the system [158].

These two measures of entanglement shall be revisited in the next chapters. We shall calculated a linear bound between random robustness and the geometrical measure of quantum discord via trace norm in Chap.4. We shall obtain, via entanglement witness, the generalized robustness of entanglement for fermionic systems of indistinguishable particles in Chap.6.

3.1.3 Entanglement witness

An approach which holds for any composed system, in any dimension and also for multipartite system is via the *entanglement witness*. As the name indicates entanglement witness are observables which characterize the entanglement of the system ([153], apud [84]). Follow the definition of the entanglement witness.

Definition 59. *A Hermitian operator $W \in \mathcal{L}(\mathbb{C}^N)$ is an entanglement witness for the entangled state ρ if:*

$$Tr(W\rho) < 0 \tag{3.22}$$
$$Tr(W\sigma) \geq 0 \quad \forall \quad \sigma \in Sep \tag{3.23}$$

As the set of separable states is convex, it is possible to obtain a limited version of the separation theorem [95].

[1] We shall discuss the distillation process in the Sec.3.1.4.

Theorem 60. *Let S be a convex set in a finite dimensional Hilbert space. Considering ρ as a point in this space, such that $\rho \notin S$. Then there exists a hyperplane that separates ρ from S.*

Given this theorem and the entanglement witness definition Horodecki *et al.* stated that: for any bipartite state there exists an entanglement witness W [81]. Therefore for any composed system there exist an observable which characterize the entanglement of the state. One important point about the entanglement witness regards about their optimality. For a given entangled state *rho* there exist a infinity of hyperplanes which witness its entanglement. Nonetheless there exist a class of optimal entanglement witness. A given entanglement witness is named *optimal* if it is tangent to the set of separable states [94]. As the entanglement witness are Hermitian matrices, they can be approach as observables, therefore can be measured directly in the lab, without perform the tomography of the state.

The optimal entanglement witness gives a notion of a distance between the state and the border of the separable set. Then this notion can be used to quantify the entanglement of the system. It is possible to define a class of measures of entanglement named *witnessed entanglement* [23,25].

Definition 61 (Witnessed entanglement)**.** *Considering a n-partite system represented by the state $\rho \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$. The witnessed entanglement is defined as:*

$$E_w(\rho) = \max\{0, -\min_{W \in \mathcal{W}}[Tr(W\rho)]\}, \tag{3.24}$$

where $\mathcal{W}$ is the set of entanglement witness of ρ.

Being the optimal entanglement witness equal W_ρ, then as it is tangent to the separable space, it has the most negative expectation value in ρ, therefore:

$$\mathrm{Tr}(W_\rho \rho) = \min_{W \in \mathcal{W}}[\mathrm{Tr}(W\rho)]. \tag{3.25}$$

As the $\mathrm{Tr}(W_\rho \rho)$ is negative for entangled states, by definition, the witnessed entanglement is positive for entangled states and is zero for separable states. The witnessed entanglement satisfies all the property of a good measure of entanglement ([30], apud [23]):

- It is invariant under local unitary operations:

$$E_w(\rho) = E_w(U \otimes \cdots \otimes U \rho U^\dagger \otimes \cdots \otimes U^\dagger), \tag{3.26}$$

  where $U \otimes \cdots \otimes U \in \mathcal{C}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$ is a local unitary operation.

- It is monotone under LOCC operations:

$$E_w(\rho) = E_w(\Gamma(\rho)), \tag{3.27}$$

  where $\Gamma \in \mathcal{C}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$ is a LOCC channel.

- It is continuous:
  For every $\epsilon \geq 0$, and density matrices ρ and σ, there exists a real number $C \geq 0$ such that holds:

$$\|\rho - \sigma\|_p \leq \epsilon \quad \Rightarrow \quad |E_w(\rho) - E_w(\sigma)| \leq C(p), \tag{3.28}$$

  where $\|\cdot\|_p$ is the Schatten-p norm, note that $C(p)$ depends on the choice of the norm.

- It is convex:

$$E_w(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E_w(\rho) + (1-\lambda)E_w(\sigma), \tag{3.29}$$

  where $\lambda \geq 0$.

The proof of these properties can be found in the reference [23].

Besides the witnessed entanglement is by itself a measure of entanglement, from it we can recover some other measures of entanglement via semi-definite programming [25]. The optimization problem of find the optimal entanglement witness is a SDP [24]. This semi definite programming is slight different fro that presented in Chap.2, here we change the condition of W be positive to be Hermitian, and the condition of $\Phi(W)$ be Hermitian to be positive. Actually there are an infinity way to write a SDP, and for each one there exist a different class of feasible points [22].

Theorem 62. *Given a bipartite state ρ, the set of entanglement witness $\mathcal{W} \in Herm$, and a linear map $\Phi : \mathcal{W} \to Pos$. We can write the primal problem of finding the optimal entanglement witness as:*

$$\begin{aligned} &\textit{maximize} && -Tr(W\rho) && (3.30)\\ &\textit{subject to} && \Phi(W) \leq B && (3.31)\\ & && Tr(\sigma W) \in Pos \quad \forall\, \sigma \in Sep && (3.32)\end{aligned}$$

The constraint $\mathrm{Tr}(\sigma W) \in Pos$ characterize W as a entanglement witness, the condition $\Phi(W) \leq B$ restricts the set of entanglement witness for an specific set where this holds for a given $B \in Pos$ and a given map Φ. For each different set of entanglement witness, which satisfies *a prior* condition, we can obtain a different measure of entanglement as the dual problem [25]. Follow three measure of entanglement, which can be written as entanglement witness, that shall be used in the next chapters:
Considering $\rho \in \mathcal{D}(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$

- Negativity:

$$\mathcal{N}(\rho) = \max\{0, \min_{0 \leq W \leq \mathbb{I}} \mathrm{Tr}(\mathbb{I} \otimes T(W)\rho)\}, \tag{3.33}$$

  where T is the transposition operation.

- Generalized Robustness:

$$\mathcal{R}_g(\rho) = \frac{1}{N} \max\{0, \min_{W \leq \mathbb{I}} \mathrm{Tr}(W\rho)\}, \tag{3.34}$$

  where $N = \dim(\mathbb{C}_{A_1} \otimes \cdots \otimes \mathbb{C}_{A_n})$.

- Random Robustness:

$$\mathcal{R}_r(\rho) = \max\{0, \min_{\mathrm{Tr}(W)=1} \mathrm{Tr}(W\rho)\}. \tag{3.35}$$

These interplay between the robustness(random and generalized) and the witnessed entanglement gave a good operational point of view for these measure of entanglement, once that now the entanglement can be calculated numerically, with a very good approximation, the entanglement of mixture states in any dimension, even multipartite [26, 27]. The problem to calculate entanglement numerically: : is that for large-dimensional Hilbert spaces it is intractable, because the entanglement, as every interesting problem in physics, is a *NP-hard problem* ([73], apud [26]). Although an algorithm can be obtained in a quasi-polinomial time for bipartite [28].

3.1.4 Entanglement cost and distillable entanglement

Another measure of entanglement for mixed state can be obtained from the quantification of entanglement for pure states. We can construct a measure of entanglement in this sense calculating the average of entanglement taken on pure states needed to form the state. The most important measure which follow this idea is named *entanglement of formation*. The entanglement of formation is interpreted as the minimal pure-states entanglement required to build the mixed state [19].

Definition 63. *Considering a quantum state $\rho \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the entanglement of formation is defined as:*

$$E_f(\rho) = min_{\xi_\rho} \sum_i p_i E(|psi_i\rangle), \tag{3.36}$$

where the optimization is performed over all ensembles $\xi_\rho = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{M}$, such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$ and $p_i \geq 0$.

As we presented below the entanglement entropy $E(|psi_i\rangle)$ is defined as:

$$E(|psi_i\rangle) = S(\mathrm{Tr}_B[|\psi_i\rangle\langle\psi_i|]), \tag{3.37}$$

where $S(\mathrm{Tr}_B[|\psi_i\rangle\langle\psi_i|])$ is the von Neumann entropy of the reduced state of $ket\psi_i$. There might exist many possible ensemble such that ρ can be constructed, in the entanglement of formation optimization process we are always looking for the ensemble with the smallest cardinality M. From the definition we can realize that for pure states the entanglement of formation is equal to the entanglement entropy. The entanglement of formation satisfies the good properties for a measure of entanglement, as the witnessed entanglement. However the most important problem of it is that: it is not easy to evaluate. Indeed the problem is related to find the minimal convex hull to form ρ in function of a nonlinear function. Although it can be calculated analytically for two qubits systems [163]. In Chap.4 we present the interplay between the entanglement of formation with the amount of classical correlations created in the purification process, in Chap.5 we shall use this approach to evaluate the entanglement of formation via the quantum discord the nonlinear optimization of quantum discord.

Quantum entanglement also enables an operational interpretation. This interpretation has two different ways: the resource required to construct a given quantum state, and the resource extracted from a quantum system. The resource here refers to the amount of copies of maximally mixture states. Then we can define the measure of this resources as a measure of entanglement, which is implicitly calculated in the limit of many copies.

The number o copies m of maximally entangled states required to construct a n copies of given state ρ, via all possible LOCC protocols, is named *entanglement cost* [19]. The entanglement cost can be written as the regularized version of the entanglement of formation [75].

Definition 64 (Entanglement cost)**.** *The number of copies of the maximally entangled states required to build the state ρ is given by:*

$$E_C(\rho) = \lim_{n\to\infty} \frac{E_f(\rho^{\otimes n})}{n}, \tag{3.38}$$

where $E_f(\rho^{\otimes n})$ is the entanglement of formation of the n copies of ρ.

The number of copies m of the maximally entangled state which can be extract from n copies of a given state ρ, via all possible LOCC protocols, is named *distillable entanglement* [19].

Definition 65 (Distillable entanglement)**.** *The distillable entanglement of a given state ρ is defines as:*

$$E_D(\rho) = \lim_{n\leftarrow\infty} \frac{m}{n}, \tag{3.39}$$

where m is the number of maximally entangled states which can be extracted from ρ in the limit of many copies.

The distillable entanglement is a very important operational measure of entanglement, because it quantifies how useful is a given quantum state, for the quantum information purpose.

The operational meaning of the entanglement cost and the distillable entanglement can be understood closer if we consider the capacity to convert many copies of a quantum state, via LOCC protocols, in maximally quantum states, and then apply the reversal process and convert the amount of copies of maximally entangled states, obtained from the distillation process, in the original quantum state. Actually it is natural to think that the amount of copies of the original state extracted in the end of the reversal process is equal to the number of copies gave as the input in the distillation process, nonetheless it is not true. Therefore, the entanglement cost and the distillable entanglement of a given state are not the same. Indeed the cost of entanglement is greater than the distillable entanglement. The point is: it is most expensive create a state with copies of maximally mixture state than we can extract from it. One example are the bound entangled state, even it is entangled it is not possible to extract any maximally mixture state[2], although it require an amount of maximally entangled states to build it.

3.2 Quantumness of Correlations

3.2.1 Classical and quantum correlated states

We shall give the an example to clarify in what sense we are characterizing classical and quantum systems. Assume a flip coin game with two distinct events described by the states $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, each with the same probability 1/2. As it is well known, it is possible to distinguish the two faces of the coin, with probability of error zero. We discussed in Chap.2 that the probability of error to distinguish two events, or two probability distributions, depends on the trace distance of the events:

$$P_E(|0\rangle\langle 0|, |1\rangle\langle 1|) = \frac{1}{2} - \frac{1}{4}|| \, |0\rangle\langle 0| - |1\rangle\langle 1| \, ||_1, \tag{3.40}$$

as the states are orthogonal $|| \, |0\rangle\langle 0| - |1\rangle\langle 1| \, ||_1 = 2$, therefore the probability of error $P_E(|0\rangle\langle 0|, |1\rangle\langle 1|) = 0$, as we expected. Now suppose we can flip a quantum coin described by the events $\{|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|\}$, with equal probability 1/2. $|\phi\rangle, |\psi\rangle \in \mathbb{C}^2$ can be arbitrary qubit states. We can consider as an example the states $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$, for this case the overlap is $\langle\phi|\psi\rangle = 1/\sqrt{2}$. The trace distance of these states is just $|| \, |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| \, ||_1 = \sqrt{2}$, then the probability of error to distinguish the events is not zero. This is what we mean with classical or quantum events. Classical events always can be distinguished, on the other hand the superposition of states in quantum mechanics creates events which cannot be perfectly distinguished. The purpose here is to characterize quantum and classical system in the context of correlations. For this we use another measure of distinguishability of events, named Jensen-Shannon divergence. For the case where there are just two probability distributions it is defined as the symmetric and smoothed version of the Shannon relative entropy, or in the quantum case the von Neumman relative entropy [98,108].

Definition 66. *The Jensen-Shannon divergence for two arbitrary events $|\psi\rangle, |\phi\rangle$ is defined as:*

$$J(|\psi\rangle, |\phi\rangle) = \frac{1}{2} S\left(\frac{|\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|}{2} \| \, |\phi\rangle\langle\phi|\right) + \frac{1}{2} S\left(\frac{|\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|}{2} \| \, |\psi\rangle\langle\psi|\right). \tag{3.41}$$

The Jensen-Shannon divergence is related to the Bures distance and it induces a true metric for pure quantum states related to the Fisher-Rao metric [91].

For the classical coin flip the Jensen-Shannon divergence will be just $J(|0\rangle, |1\rangle) = 1$. On the other hand, for the quantum coin flip with states $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$, it will be $J(|\phi\rangle, |\psi\rangle) = \sqrt{2}$. If the Jensen-Shannon divergence was a true measure of distance, we expect that this distance would be larger for events which could be always distinguished. On the other hand, the Jensen-Shannon divergence for two events is related to the mutual information between a random variable and another register with distinguishable events. For our two arbitrary events $|\psi\rangle, |\phi\rangle$ [108]:

$$J(|\psi\rangle, |\phi\rangle) = I(R : E), \tag{3.42}$$

[2]The name come from this property.

where R represents the register and E represents the events. For two arbitrary events $|\psi\rangle, |\phi\rangle \in \mathbb{C}_E$ we can create another state $\rho_{RE} \in \mathcal{D}(\mathbb{C}_R \otimes \mathbb{C}_E)$, which represents the possible events and the register, it is written as:

$$\rho_{RE} = \frac{1}{2}|0\rangle\langle 0|_R \otimes |\phi\rangle\langle\phi|_E + \frac{1}{2}|1\rangle\langle 1|_R \otimes |\psi\rangle\langle\psi|_E . \tag{3.43}$$

Then, coming back to the coin flip example, for the classical coin flip case it is: $\rho^c_{RE} = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$, with mutual information $I(R:E)_{\rho^c_{RE}} = 1$, and for the quantum coin flip, described above, the state will be $\rho^q_{RE} = \frac{1}{2}|0\rangle\langle 0| \otimes |\phi\rangle\langle\phi| + \frac{1}{2}|1\rangle\langle 1| \otimes |\psi\rangle\langle\psi|$, where for $|\phi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\psi\rangle = |1\rangle$, and the mutual information is $I(R:E)_{\rho^q_{RE}} = \sqrt{2}$. As the mutual information is a measure of correlations between two probability distributions, we realize that there are more correlations between the register and the events which are not distinct than the events which are completely distinguishable. However we know that two binary classical distributions cannot share more then one bit, in other words, their mutual information cannot be greater than one. As the correlations between the quantum coin events and the register are bigger than one, it means that there are correlations beyond the classical ones, therefore they are quantum correlated, even though they are not entangled by definition.

Indeed there exists another kind of quantum correlation for which the quantum entanglement is a subset. This is the kind of quantum correlations which we are interested. This was named as *quantumness of correlations* [115,117], which characterizes the degree of quantumness of the system from the correlation point of view.

As one can perceive in the last example, we can change from ρ^c_{ER} to ρ^q_{ER} only applying local operations, and it is not acceptable that correlations can be created without interaction, therefore quantumness of correlations expresses a quantum character of classical correlations [115].

The classical correlated state ρ^c_{ER}, described above, is just an example which is named *classical-classical*, because it is strictly classically correlated. The quantum correlated state ρ^q_{ER} is named *classical-quantum*. In general we can define classical correlated states as the states which remain undisturbed by a local measurements, in other words, there exists a local projective measurement which the state remains the same [76,115,117].

Definition 67. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, it is strictly classical correlated if there exists a local projective measurement $\Pi \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_B)$ with elements $\{\Pi^A_l \otimes \Pi^B_k\}_{k,l}$ such that the post-measurement state is equal to the input state:*

$$\Pi(\rho_{AB}) = \sum_{k,l} \Pi^A_l \otimes \Pi^B_k \rho_{AB} \Pi^A_l \otimes \Pi^B_k = \rho_{AB}, \tag{3.44}$$

therefore $\rho_{AB} = \sum_{k,l} p_{k,l} \Pi^A_l \otimes \Pi^B_k$.

The states which satisfy definition 67 are called *classical-classical states*, being ρ^c_{ER} an example. The states such that there exists a local measurement which acts just on one subsystem and the state remains unchanged is also classical correlated in relation with that part and these states are named classical-quantum or quantum-classical, depending on which subsystem it is invariant under the projective measurement. In other words, the state ρ_{AB} is classical-quantum if there exists a projective measurement $\Pi_A \in \mathcal{P}(\mathbb{C}_A, \mathbb{C}_A)$ such that:

$$\Pi_A \otimes \mathbb{I}_B(\rho_{AB}) = \rho_{AB} = \sum_k p_k |a_k\rangle\langle a_k| \otimes \rho_k. \tag{3.45}$$

We name the set of classical-classical states as Ω_{cc} and the space of classical-quantum states as Ω_{cq}[3]. These sets of classically correlated states live inside the subspace of the separable states and $\Omega_{cc} \subseteq \Omega_{cq}$. As the set of classical correlated states is composed by block diagonal matrices

[3] Often we shall say just Ω or the set of classical correlated states because it is general and encompasses all the sub definitions for these states.

it is not convex, because we cannot guarantee that the convex combination of block diagonal matrices remains block diagonal. As the identity matrix is block diagonal, or just diagonal, this set is connected by the maximal mixture state. Finally, this set is a *thin set* or in a terminology used in the area of quantum information it has *null measure* [61].

3.2.2 Quantum discord

The amount of classical correlations in a quantum state is measured by the capacity to extract information locally. As the measurement process is a classical statistical inference, it can be measured by the amount of correlations that remains in the system after a local measurement.

Definition 68. *For a bipartite density matrix $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the classical correlations between A and B can be quantified by the amount of correlations which can be extracted via local measurements [4]:*

$$J(A:B)_{\rho_{A:B}} = \max_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})}, \tag{3.46}$$

where the optimization is taken over the set of local measurement maps $\mathbb{I}\otimes\mathcal{B} \in \mathcal{P}(\mathbb{C}_{AB}, \mathbb{C}_{AX})$ and $\mathbb{I}\otimes\mathcal{B}(\rho_{AB}) = \sum_x p_x \rho_x^A \otimes |b_x\rangle\langle b_x|$ is a quantum-classical state in the space $\mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_X)$.

Originally H. Ollivier and W. Zurek [115] defined this expression with the optimization restricted to projective measurements. Independently, L. Henderson and V. Vedral [76] defined the optimization of the classical correlations over general measurements. As the mutual information quantifies the total amount of correlations in the state, it is possible to define a measure of quantum correlations as the difference between the total of correlations in the system, quantified by the mutual information, and only the classical correlations, measured by Eq.3.46. This measure of quantumness of correlations is named *quantum discord*:

Definition 69. *The quantum discord $D(A:B)_{\rho_{AB}}$ of a state ρ_{AB} is defined as:*

$$D(A:B)_{\rho_{AB}} = I(A:B)_{\rho_{AB}} - J(A:B)_{\rho_{AB}}, \tag{3.47}$$

where $I(A:B)_{\rho_{AB}}$ is the von Neumann mutual information.

The quantum discord quantifies the amount of information which cannot be accessed via local measurements, therefore it measures the quantumness which is shared between A and B that cannot be recovered via a classical statistical inference process. We decided to use the formalism of quantum channels to describe the measurement on the quantum discord, although Eq.3.46 can be written as originally defined in the seminal papers [76, 115], we enunciate it as proposition:

Proposition 70. *The Eq.3.46 can be rewritten as:*

$$J(A:B)_{\rho_{A:B}} = \max_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} \left\{ S(\rho_A) - \sum_x p_x S(\rho_x^A) \right\}. \tag{3.48}$$

where $p_x\rho_x^A = Tr_b[\mathbb{I}\otimes B_x\rho_{AB}]$ and the $\{B_x\}_x$ are the elements of the POVM $\mathcal{B} \in \mathcal{P}(\mathbb{C}_B, \mathbb{C}_X)$.

Proof. Given a measurement map $\mathbb{I}\otimes\mathcal{B}$, such that

$$\rho_{AX} = \mathbb{I}\otimes\mathcal{B}(\rho_{AB}) = \sum_x p_x\rho_x^A \otimes |b_x\rangle\langle b_x|, \tag{3.49}$$

where $p_x\rho_x^A = Tr_B[\mathbb{I}\otimes B_x\rho_{AB}]$, and the mutual information will be:

$$I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} = S(\rho_A) + S(\rho_X) - S(\rho_{AX}), \tag{3.50}$$

[4] In the definition we are not taking care to label the function $J(A:B)$ in which subsystem the measurement is being applied, although we can specify if necessary.

then $S(\rho_X) = H(X) = \sum_x p_x \log p_x$ and:

$$S(\rho_{AX}) = H(X) - \sum_x p_x S(\rho_x^A), \tag{3.51}$$

which is the Shannon entropy for block diagonal states. Substituting $S(\rho_X)$ and Eq.3.51 in the Eq.3.50 we prove the proposition. □

The optimization of quantum discord is a NP-hard problem, in the sense that to calculate quantum discord the running time for any algorithm grows exponentially with the dimension of the measured system [85]. A general analytical solution for quantum discord is not known, nor a criterion for a giving POVM to be optimal, for a specific state. Nonetheless there are some analytic expressions for some specific states [6,67,100,101].

It is natural from Eq.3.47 a generalization of quantum discord to the case in which the measurement is applied locally on both subsystems.

Definition 71. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ the quantum discord over measurements on both systems is:*

$$D(A:B)_{\rho_{AB}} = \min_{\mathcal{A}\otimes\mathcal{B}\in\mathcal{P}} \left\{ I(A:B)_{\rho_{AB}} - I(A:B)_{\mathcal{A}\otimes\mathcal{B}(\rho_{AB})} \right\}, \tag{3.52}$$

where $\mathcal{A} \in \mathcal{P}(\mathbb{C}_A, \mathbb{C}_Y)$ and $\mathcal{B} \in \mathcal{P}(\mathbb{C}_B, \mathbb{C}_X)$.

This generalization of quantum discord was first discussed in [130] in the context of the no-local-broadcast theorem. This definition is often named WPM-discord, because it was also studied by S. Wu *et. al* [165]. It was also approached restricted to projective measurements by some authors [68,110,137]. Analogous to the generalization done above, it is possible to propose a measure of quantum discord for multipartite systems. First let us understand the mutual information for multipartite systems. The mutual information $I(A:B)$ quantifies the number of bits that part A must send to B to destroy the correlations between them. In this same way a multipartite and nonpairwise version of mutual information $I(A_1 : ... : A_N)$ can be defined as the number of bits which A_1 must send to destroy its correlations with the others $A_2, ..., A_N$ subsystems ($I(A_1 : A_2, ..., A_N)$), summed what A_2 must spend to destroy its correlations with the remaining subsystems $I(A_2 : A_3, ..., A_N|A_1)$, so forth, until all the systems be uncorrelated [71].

Definition 72. *Given a multipartite state $\rho_{A_1,...,A_N}$, composed by N subsystems. The mutual information which represents the above process can be written as:*

$$I(A_1 : ... : A_N) = I(A_1 : A_2, ..., A_N) + I(A_2 : A_3, ..., A_N|A_1) + \cdots + I(A_N|A_1, \ldots, A_{N-1}). \tag{3.53}$$

where $I(A:B|C)_{\rho_{ABC}} = S(A|C)_{\rho_{ABC}} + S(B|C)_{\rho_{ABC}} - S(AB|C)_{\rho_{ABC}}$ is the conditional mutual information for the state ρ_{ABC}.

The multipartite mutual information has the same properties of the pairwise mutual information. It comes from the fact that both of them can be written as the relative entropy between the state and the product of the marginals. From the operational point of view it is the same that the amount of information needed to destroy the correlations between all the systems, measured in bits.

Proposition 73. *Given the relative entropy $S(\rho_{A_1,...,A_N}||\rho_{A_1} \otimes \cdots \otimes \rho_{A_N})$, for the multipartite state $\rho_{A_1,...,A_N}$, the multipartite mutual information can be rewritten as:*

$$I(A_1 : ... : A_N) = S(\rho_{A_1,...,A_N}||\rho_{A_1} \otimes \cdots \otimes \rho_{A_N}) = \sum_i S(\rho_{A_i}) - S(\rho_{A_1,...,A_N}), \tag{3.54}$$

where $\rho_i = Tr_{\widehat{A_i}}[\rho_{A_1,...,A_N}]$ is the state of the i^{th} subsystems, the notation $\widehat{A_i}$ means that the trace is taken over all subsystems except the system A_i.

Proof. Given the definition of the conditional entropy and using the same tricks used in Chap.2 in the proof of proposition 32:

$$S(\rho_{A_1,\dots,A_N}||\rho_{A_1}\otimes\cdots\otimes\rho_{A_N}) = -S(\rho_{A_1,\dots,A_N}) - \mathrm{Tr}[\rho_{A_1,\dots,A_N}\log\rho_{A_1}\otimes\cdots\otimes\rho_{A_N}] \quad (3.55)$$
$$= -S(\rho_{A_1,\dots,A_N}) + S(\rho_{A_1}) + \cdots + S(\rho_{A_N}). \quad (3.56)$$

For the mutual information we will prove for tripartite systems, although it is a particular case, we can realize from it that the result is valid for any multipartite system. From the chain rule for the mutual information:

$$I(A:BC) = I(A:B) + I(A:B|C), \quad (3.57)$$

we can rewrite Eq.3.53 as:

$$I(A:B:C) = I(A:BC) + I(B:C) = S(A) + S(B) + S(C) - S(ABC). \quad (3.58)$$

□

Given the definition of the multipartite mutual information we are close to define a multipartite version of the quantum discord, we first should define the multipartite measurement map $\mathcal{M}_K \in \mathcal{P}(\mathbb{C}_{A_1,\dots,A_k}, \mathbb{C}_{X_1,\dots,X_k})$, that acts locally on the fist k subsystems of the composed system describe by $\rho_{A_1,\dots,A_N}$. Then the post-measurement state is defined as:

$$\mathcal{M}_K\otimes\mathbb{I}_{A_{k+1,\dots,N}}(\rho_{A_1,\dots,A_N}) = \sum_{f(x)}\mathrm{Tr}_{A_1,\dots,A_k}[E_{f(x)}\otimes\mathbb{I}_{A_{k+1,\dots,N}}\rho_{A_1,\dots,A_N})]\otimes|f(x)\rangle\langle f(x)|, \quad (3.59)$$

where $|f(x)\rangle = |x_1\rangle\cdots|x_k\rangle$ and $E_{f(x)} = E_{x_1}\otimes\cdots\otimes E_{x_k}$. Analogous to the bipartite case the quantum discord is defined as the smallest difference between the total correlations and the remained correlations after the local measurement.

Definition 74. *For a given N-partite system, described by the state $\rho = \rho_{A_1,\dots,A_N}$, the multipartite quantum discord can be defined as [130]:*

$$D(A_1:\dots:A_N)^K_\rho = \inf_{\mathcal{M}_K\in\mathcal{P}}\left\{I(A_1:\dots:A_N)_\rho - I(X_1:\dots:X_k:A_{k+1}:\dots:A_N)_{\mathcal{M}_K\otimes\mathbb{I}(\rho)}\right\}, \quad (3.60)$$

where the superscript K on the quantum discord is just to indicate that the measurements will be taken over the first k subsystems.

Rulli *et. al* [137] discussed a definition of multipartite quantum discord which follows the same idea, although their definition is restricted to projective measurements. The properties of quantum discord, shall be calculated for bipartite systems, however they remains true for multipartite systems [130].

Properties of quantum discord

For a bipartite quantum state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A\otimes\mathbb{C}_B)$, follow the properties of quantum discord:

Proposition 75 (i)**.** *Quantum discord is not symmetric:*

$$D(A:B)_{\rho_{AB}} \neq D(B:A)_{\rho_{AB}}. \quad (3.61)$$

Proof:(i). Given a measurement map $\mathcal{M}\in\mathcal{P}(\mathbb{C}_X,\mathbb{C}_Y)$ in general

$$\mathbb{I}\otimes\mathcal{M}(\rho_{AB}) \neq \mathcal{M}\otimes\mathbb{I}(\rho_{AB}), \quad (3.62)$$

which implies that the mutual information $I(A:Y)\neq I(Y:B)$. □

For the quantum discord which the measurements are applied on both subsystems it is clear that it is symmetric, by this property it is sometimes named symmetric quantum discord.

Proposition 76 (ii). *Quantum discord is non-negative:*

$$D(A:B)_{\rho_{AB}} \geq 0. \tag{3.63}$$

Proof:(ii). As the mutual information is monotone decreasing under LOCC maps:

$$I(A:B)_{\rho_{AB}} \geq I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})}, \tag{3.64}$$

where $\mathcal{B} \in \mathcal{P}(\mathbb{C}_A, \mathbb{C}_Y)$. □

Proposition 77 (iii). *Quantum discord vanishes if and only if the state is classical correlated.*

Proof:(iii). The easiest way to prove this property is via the activation protocol which we will present in next chapter [129]. Given the theorem 2 of Ref. [150] that the quantum discord of ρ_{AB} is equal the minimal partial distillable entanglement between the measured subsystem and the measurement apparatus:

$$D(A:B)_{\rho_{AB}} = \min_{U} P_{E_D}(\mathbb{I}_A \otimes U_{BE}(\rho_{AB} \otimes |0\rangle\langle 0|_E)\mathbb{I}_A \otimes U_{BE}^{\dagger}), \tag{3.65}$$

where U_{BE} are unitary evolutions which characterize a projective measurement on system B, therefore $\mathrm{Tr}_E[U_{BE}\sigma_{BE}U_{BE}^{\dagger}] = \sum_x \Pi_x \sigma_B \Pi_x$. The partial distillable entanglement $P_{E_D}(\rho_{ABE}) = E_D(\rho_{AB:E}) - E_D(\rho_{B:E})$ quantifies the entanglement lost if the part A is ignored [150]. As the activation protocol tells that the entanglement created between the system and the measurement apparatus is zero if and only of the state is classical correlated [129], the discord will be zero when there exists a measurement unitary evolution $U_{BE}^{\dagger}$ such that $P_{E_D}(\mathbb{I}_A \otimes U_{BE}(\rho_{AB} \otimes |0\rangle\langle 0|_E)\mathbb{I}_A \otimes U_{BE}^{\dagger}) = 0$, therefore it will be zero if and only if the state is classical correlated. □

A. Datta proved this statement from the strong sub-additivity of the von Neumann entropy [43].

Proposition 78 (iv). *Quantum discord is invariant under local unitary operations:*

$$D(A:B)_{\rho_{AB}} = D(A:B)_{(U_A\otimes U_B)\rho_{AB}(U_A^{\dagger}\otimes U_B^{\dagger})}. \tag{3.66}$$

Proof:(iv). Suppose the POVM $\mathcal{B}$ with elements $\{B_x\}_x$ is the optimal in the calculation of quantum discord $D(A:B)_{\rho_{AB}}$, hence the POVM $\tilde{\mathcal{B}}$ with elements $\{U_B^{\dagger}B_xU_B\}_x$ will be the optimal for $D(A:B)_{(U_A\otimes U_B)\rho_{AB}(U_A^{\dagger}\otimes U_B^{\dagger})}$. Both expressions are the same because

$$p_xU_A\rho_x^AU_A^{\dagger} = \mathrm{Tr}_B[\mathbb{I}\otimes B_x(U_A\otimes U_B)\rho_{AB}(U_A^{\dagger}\otimes U_B^{\dagger})] = \mathrm{Tr}_B[\mathbb{I}\otimes(U_B^{\dagger}B_xU_B)(U_A\otimes\mathbb{I})\rho_{AB}(U_A^{\dagger}\otimes\mathbb{I})]. \tag{3.67}$$

Then, as the von Neumann entropy is invariant under unitary operations: $S(U_A\rho_x^AU_A^{\dagger}) = S(\rho_x^A)$ and $S(U_A\rho_AU_A^{\dagger}) = S(\rho_A)$. □

Proposition 79 (v). *Quantum discord does not necessary decrease under local maps which acts on the measured subsystem, however it is contractive under local maps on the non measured subsystem [127].*

Proof:(v). The first statement can be checked with an example, and it was given in last section. Suppose a local map $\Phi \in \mathcal{C}(\mathbb{C}_B)$ which acts on the following way on the orthogonal basis $\{|0\rangle, |1\rangle\}$ $\Phi(|0\rangle) = (|0\rangle + |1\rangle)/\sqrt{2}$ and $\Phi(|1\rangle) = |1\rangle$. Then given the classical-classical state $\rho_{AB}^{cc} = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$ after the action of the map it will be:

$$\Phi(\rho_{AB}^{cc}) = (|0+\rangle\langle 0+| + |11\rangle\langle 11|)/2, \tag{3.68}$$

for $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. As the state after the action of the map is a classical-quantum state, therefor it is possible to create quantum discord for measurements on the subsystem B.

The second statement comes from the fact that the mutual information is monotone decreasing. Given the definition of the quantum discord:

$$D(A:B) = \inf_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} \left\{ I(A:B)_{\rho_{AB}} - I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} \right\}, \tag{3.69}$$

Suppose a local isometry $V \in \mathcal{U}(\mathbb{C}_B, \mathbb{C}_{XR})$, such that $\mathrm{Tr}_R[\mathbb{I}_A \otimes V\rho_{AB}\mathbb{I}_A \otimes V^\dagger] = \mathbb{I}\otimes\mathcal{B}(\rho_{AB})$. As the mutual information is invariant under local isometries $I(A:B)_{\rho_{AB}} = I(A:XR)_{\rho_{AXR}}$ and $I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} = I(A:X)_{\rho_{AXR}}$, therefore:

$$I(A:B)_{\rho_{AB}} - I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} = I(A:XR)_{\rho_{AXR}} - I(A:X)_{\rho_{AXR}} = I(A:R|X), \tag{3.70}$$

where in the right hand side we used the definition of the conditional mutual information. As the conditional mutual information is contractive under the action of local *PPT* maps on A and R, this proves the property. □

This property of quantum discord also can be proved via the activation protocol [129] using the monotonicity of the entanglement measures [150].

Proposition 80 (vi)**.** *Quantum discord is upper bounded by the von Neumann entropy of the measured subsystem [96]:*

$$D(A:B)_{\rho_{AB}} \leq S(B). \tag{3.71}$$

Proof:(vi). Given a purification of ρ_{AB} as $\rho_{ABE} = |\psi\rangle\langle\psi|_{ABE}$ and the post-measurement state[5]

$$\sigma_{ABE} = \mathbb{I}_{AE} \otimes \mathcal{M}(\rho_{ABE}) = \sum_x p_x |\psi_x\rangle\langle\psi_x|_{AE} \otimes |b_x\rangle\langle b_x|, \tag{3.72}$$

for a given measurement map $\mathcal{M} \in \mathcal{P}(\mathbb{C}_B, \mathbb{C}_X)$, which represents a POVM $\mathcal{B}$ with rank-1 elements, then:

$$\begin{aligned}
I(A:B)_{\sigma_{AB}} - I(B:E)_{\sigma_{BE}} &= \left(S(\rho_A) - \sum_x p_x S(\rho_x^A)\right) - \left(S(\rho_E) - \sum_x p_x S(\rho_x^E)\right) &(3.73)\\
&= S(\rho_A) - S(\rho_E) &(3.74)\\
&= S(\rho_A) - S(\rho_{AB}), &(3.75)
\end{aligned}$$

where $\rho_x^A = \rho_x^C = \mathrm{Tr}_A[|\psi_x\rangle\langle\psi_x|_{AE}]$. As the classical correlations, by definition, is $J(A:B)_{\rho_{AB}} \geq I(A:B)_{\sigma_{AB}}$, hence the quantum discord is:

$$\begin{aligned}
D(A:B)_{\rho_{AB}} &= I(A:B)_{\rho_{AB}} - J(A:B)_{\rho_{AB}} &(3.76)\\
&\leq I(A:B)_{\rho_{AB}} - I(A:B)_{\sigma_{AB}} &(3.77)\\
&= I(A:B)_{\rho_{AB}} - S(\rho_A) - S(\rho_{AB}) &(3.78)\\
&= S(\rho_B). &(3.79)
\end{aligned}$$

□

It was conjectured that the quantum discord is upper bounded by the entropy of both marginals [105], however the it was disproved that by N. Li and S. Luo [96].

Proposition 81 (vii)**.** *The mutual information $I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})}$ is convex on the POVMs.*

[5] A detailed explanation about the action of measurement maps on the purified state shall be done in the Section 4.1.

Proof:(vii). Consider a POVM $\mathcal{B} \in \mathcal{P}(\mathbb{C}_B, \mathbb{C}_X)$ which can be written as convex combination of a set of POVMs $\tilde{\mathcal{B}}_k \in \mathcal{P}(\mathbb{C}_B, \mathbb{C}_{Y_k})$:

$$\mathcal{B} = \sum_k q_k \tilde{\mathcal{B}}_k. \tag{3.80}$$

As a quantum channel is a linear map, applying the local measurement map of $\mathcal{B}$ on a bipartite state ρ_{AB}:

$$\mathbb{I} \otimes \mathcal{B}(\rho_{AB}) = \sum_k q_k \mathbb{I} \otimes \tilde{\mathcal{B}}_k(\rho_{AB}), \tag{3.81}$$

where $\sum_k q_k = 1$. The mutual information of the post-measurement state can be written as:

$$\begin{aligned} I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} &= S(\mathbb{I} \otimes \mathcal{B}(\rho_{AB}) \| \rho_A \otimes \mathcal{B}(\rho_B)) && (3.82)\\ &= S(\sum_k q_k \mathbb{I} \otimes \tilde{\mathcal{B}}_k(\rho_{AB}) \| \sum_k q_k \rho_A \otimes \tilde{\mathcal{B}}_k(\rho_B)) && (3.83)\\ &\leq \sum_k q_k S(\mathbb{I} \otimes \tilde{\mathcal{B}}_k(\rho_{AB}) \| \rho_A \otimes \tilde{\mathcal{B}}_k(\rho_B)) && (3.84)\\ &= \sum_k q_k I(A:Y_k)_{\mathbb{I}\otimes\tilde{\mathcal{B}}_k(\rho_{AB})}. && (3.85) \end{aligned}$$

In Eq.3.83 we used the linearity of the trace, in Eq.3.84 we used the joint convexity property of the relative entropy. □

This statement implies that the non optimized quantum discord is a concave function. The way we proved the proposition 81 is original of this thesis, and in the opinion of the author is clearer than the proof performed in the literature. Another original way to prove it can be obtained from the concavity of the conditional mutual information $I(A:E|X)$ on the variable X, which represents the after measured state. The proof of this statement also can be find in the literature on the A. Datta phd thesis [42], where the concavity of the quantum discord can be obtained from the concavity of the conditional entropy.

Corolary 82 (vii)**.** *There exists an optimal POVM with rank-1 elements.*

Proof: Corollary (vii). Given a POVM $\mathcal{B}$ whose elements $\{B_x\}_x$ are not rank-1, it is possible to decompose each one in its eigenbasis $B_x = \sum_z C_{xz}$, where the operators C_{xz} are rank-1. In this way we can rewrite the POVM $\mathcal{B} = \bigoplus_z p_z \mathcal{C}_z$, where $\sum_z p_z = 1$. As the mutual information $I(A:X)_{\mathbb{I}\otimes\mathcal{M}(\rho_{AB})}$ is convex on the POVMs:

$$I(A:X)_{\mathbb{I}\otimes\mathcal{B}(\rho_{AB})} \leq \sum_z p_z I(A:X)_{\mathbb{I}\otimes\mathcal{C}_z(\rho_{AB})} \leq I(A:X)_{\mathbb{I}\otimes\mathcal{C}_z(\rho_{AB})}. \tag{3.86}$$

Suppose the POVM $\mathcal{B}$ optimizes the classical correlations, then there exist one value of z such that a $\mathcal{C}_z$ also maximizes the classical correlations. □

Proposition 83 (viii)**.** *For bipartite pure states the quantum discord is equal to the entropy of entanglement [76]:*

$$D(A:B)_{\Psi_{AB}} = S(\rho_A), \tag{3.87}$$

where $\Psi_{AB} = |\psi\rangle\langle\psi|_{AB}$ and $\rho_A = Tr_B[\Psi_{AB}]$.

Proof. Given the mutual information for pure states:

$$I(A:B)_{\Psi} = 2S(\rho_A), \tag{3.88}$$

the quantum discord will be:

$$D(A:B)_{\Psi_{AB}} = S(\rho_A) + \inf_{\mathbb{I}\otimes\mathcal{B}\in\mathcal{P}} \{S(\mathbb{I} \otimes \mathcal{B}[\Psi_{AB}]) - S(\mathcal{B}[\rho_B])\}. \tag{3.89}$$

If we find a POVM such that the difference $S(\mathbb{I}\otimes\mathcal{B}[\Psi_{AB}]) - S(\mathcal{B}[\rho_B]) = 0$ it of course will be the optimum POVM. This optimum POVM is a dephasing channel in the Schmidt basis. Given the Schmidt decomposition of the state: $|\psi\rangle = \sum_i c_i |a_i\rangle |b_i\rangle$, the state after the local dephasing on B:

$$\mathbb{I}\otimes\Pi_B[\Psi] = \sum_{ij} c_i c_j |a_i\rangle\langle a_j| \otimes \Pi_B[|b_i\rangle\langle b_j|] = \sum_i c_i c_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i| . \tag{3.90}$$

As this state is classical-classical:

$$S(\mathbb{I}\otimes\mathcal{B}[\Psi_{AB}]) = S(\mathcal{B}[\rho_B]). \tag{3.91}$$

□

3.2.3 Geometrical approach

In this section we shall give a geometrical point of view for quantumness of correlations. As quantum entanglement, the quantumness of correlations can be quantified geometrically by the distance between the state and the set of states without quantum correlations. As the space of states in quantum mechanics is not a flat space there are many different kinds of measures of distances [14]. Independently T. Debarba *et al.* [47] and T. Nakano *et al.* [113] proposed that the geometrical quantifiers of quantumness of correlations can be via Schatten-p norm, from this measure it is possible to define other measures via Hilbert-Schmidt distance named geometrical quantum discord [40, 104] and via trace distance named 1-norm geometrical quantum discord [47, 113]. It is also possible to measure quantumness of correlations via Fidelity or Bures distance [4, 148]. Another measure of quantumness based on distance is the via the relative entropy [112, 117]. Actually the distance between the state and the set of classical correlated states is not the only way to quantify quantumness of correlations geometrically, another way is based on the disturbance created in the state after a local measurement [102], therefore distances between the state and the disturbed state also quantifies quantum correlations. However we shall focus on the measure of distance based on the Schatten-p norm, notwithstanding we shall dedicate a whole section to discuss the measure of quantumness of correlation based on the relative entropy.

In general, for a state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A, \mathbb{C}_B)$, a geometrical measure of quantum discord can be defined as the distance between ρ_{AB} and the closest classical correlated state. An interesting norm to describe the quantum discord is the Schatten-p norm, as discussed in the Chap.2, carries in its definition the trace distance and the Hilbert-Schmidt distance.

Definition 84. *The geometrical quantum discord for any Schatten-p norm can be defined as [47, 113]:*

$$D_p(\rho_{AB}) = \inf_{\xi_{AB}\in\Omega_{QC}} ||\rho_{AB} - \xi_{AB}||_p^p, \tag{3.92}$$

where $||A||_p = Tr[(A^\dagger A)^{p/2}]^{1/p}$ is the Schatten-p norm.

The Schatten-p geometrical discord will be zero if and only if the state ρ_{AB} is inside the set of classical correlated states.

For $p = 2$ the Schatten-p norm represents the Hilbert-Schmidt norm and the quantum discord is named *geometrical discord* [40]:

$$D_2(\rho_{AB}) = \inf_{\xi_{AB}\in\Omega_{QC}} ||\rho_{AB} - \xi_{AB}||_2^2. \tag{3.93}$$

This measure of quantum discord can be calculated analytically for general bipartite state [104] and has friendly expressions for $2\otimes 2$ systems [40, 67]. S. Luo and S. Fu showed that the geometrical quantum discord is equal to the local disturbance [104]:

$$D_2(\rho_{AB}) = \inf_{\Pi_B\in\mathcal{P}(\mathbb{C}_B)} ||\rho_{AB} - \mathbb{I}_A\otimes\Pi_B(\rho_{AB})||_2^2, \tag{3.94}$$

where now the optimization is taken over local projective measurement maps $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$. Using the disturbance is possible to calculate the quantum discord analytically for any pure state, Bell diagonal state and $2 \otimes n$-dimensional states [103]. The geometrical measure of discord was also studied in the context of the quantum computation protocols [40, 120]. It was also measured experimentally [39, 147]. The geometrical quantum discord has a special interesting because it is very operational. It is just the smallest difference between the purity of the state ($\text{Tr}[\rho^2]$) and the purity of the local dephased state ($\text{Tr}[(\mathbb{I} \otimes \Pi(\rho))^2]$). We state it as a proposition [103].

Proposition 85. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ the geometrical discord Eq.3.94 can be written as:*

$$D_2(\rho_{AB}) = \inf_{\Pi_B \in \mathcal{P}(\mathbb{C}_B)} \left\{ Tr[\rho_{AB}^2] - Tr[\mathbb{I}_A \otimes \Pi_B(\rho_{AB})^2] \right\}, \tag{3.95}$$

where $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$ is a dephasing channel.

Proof. The square of the Hilbert-Schmidt norm:

$$||\rho_{AB} - \mathbb{I}_A \otimes \Pi_B(\rho_{AB})||_2^2 = \text{Tr}[\rho_{AB}^2] - 2\text{Tr}[\rho_{AB} \cdot \mathbb{I}_A \otimes \Pi_B(\rho_{AB})] + \text{Tr}[\mathbb{I}_A \otimes \Pi_B(\rho_{AB})^2]. \tag{3.96}$$

We can write a general bipartite state as $\rho_{AB} = \sum_{ij} O_{ij}^A \otimes |i\rangle\langle j|$, the local dephasing channel acts on ρ_{AB} as $\mathbb{I} \otimes \Pi(\rho_{AB}) = \sum_{ij} O_{ij}^A \otimes \Pi(|i\rangle\langle j|)$, where $\Pi(|i\rangle\langle j|) = \sum_k \delta_{ik}\delta_{jk} |k\rangle\langle k|$, then:

$$\begin{aligned}
\text{Tr}[\rho_{AB}\mathbb{I} \otimes \Pi(\rho_{AB})] &= \text{Tr}[\sum_{ij} O_{ij}^A \otimes |i\rangle\langle j| \cdot \sum_{kl} O_{kl}^A \otimes \Pi(|k\rangle\langle l|)] & (3.97)\\
&= \sum_{ij}\sum_{kl} \text{Tr}[\rho_{ij}^A \rho_{kl}^A]\text{Tr}[\Pi(|i\rangle\langle j|)\, |k\rangle\langle l|] & (3.98)\\
&= \sum_{ij}\sum_{kl} \text{Tr}[\rho_{ij}^A \rho_{kl}^A]\text{Tr}[\Pi(|i\rangle\langle j|)\Pi(|k\rangle\langle l|)] & (3.99)\\
&= \text{Tr}[\Pi(\rho_{AB})\Pi(\rho_{AB})]. & (3.100)
\end{aligned}$$

Where we used in Eq.3.99 that $\text{Tr}[\Pi(\sigma)\gamma] = \text{Tr}[\Pi(\sigma)\Pi(\gamma)]$, for two square matrices $\sigma, \gamma : \mathbb{C}^d \to \mathbb{C}^d$, and in Eq.3.100 we reordered the terms to get ρ_{AB} again. □

Although the geometrical discord is an operational measure for quantumness of correlation it does not satisfy the good conditions to be a quantifier of quantum correlation [29]. The commonly acceptable properties for quantumness of correlations are the properties 76-79 of quantum discord. The problem with the geometrical discord is that it does not satisfy the second statement on property 79, i.e, it can increase under local channels on the non measured subsystem [127]. It comes from the fact that the Hilbert-Schmidt norm is not monotonic decreasing under PPT maps[6]. A simple example was given by Marco Piani [127]. Suppose a bipartite state ρ_{AB} and a local projective measurement $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$ on B, then the Hilbert-Schmidt distance between the state and the dephased state will be:

$$D(AB, AB^{\Pi})_{\rho_{AB}} = ||\rho_{AB} - \mathbb{I}_A \otimes \Pi_B(\rho_{AB})||_2. \tag{3.101}$$

Now suppose a channel $\Gamma_A \in \mathcal{C}(\mathbb{C}_A, \mathbb{C}_A \otimes \mathbb{C}_E)$ which acts on A just appending an ancilla σ with arbitrary dimension:

$$\Gamma_A \otimes \mathbb{I}_B(\rho_{AB}) = \rho_{A'B} = \rho_{AB} \otimes \sigma_E. \tag{3.102}$$

The Hilbert-Schmidt distance between $\rho_{AB} \otimes \sigma_E$ and the dephased state on B:

$$D(A'B, A'B^{\Pi})_{\Gamma_A \otimes \mathbb{I}_B(\rho_{AB})} = ||\rho_{AB} \otimes \sigma_E - \mathbb{I}_A \otimes \Pi_B(\rho_{AB}) \otimes \sigma_E||_2 = ||\rho_{AB} \otimes - \mathbb{I}_A \otimes \Pi_B(\rho_{AB})||_2 \cdot ||\sigma||_2, \tag{3.103}$$

[6]This was discussed in the context of quantum entanglement when arise the question if Hilbert-Schmidt norm was a good measure for quantum entanglement [118].

however $||\sigma||_2 = \sqrt{\mathrm{Tr}[\sigma^2]}$, which will be equal to 1 only if the ancilla is a pure state. As we can apply local operations on the ancilla, then applying a purification on the ancillary system it can increase the geometrical quantum discord just applying local operations, which means that we are increasing the amount of total correlations without creating any correlations in the system [127]. The problems of the geometric discord based on Hilbert-Schmidt distance were solved redefining a measure of quantum correlations in function of the geometric discord. This function keeps some entropic properties of the quantum discord [154].

The Schatten-1 norm was first proposed by T. Debarba *et al.* to obtain a linear bound between geometrical discord via trace distance and random robustness of entanglement [47,48]. Independently other authors proposed it in different contexts [113, 154]. Indeed the Schatten-1 norm is the only norm which is an acceptable measure for quantumness of correlations [122]. By the last example for the state $\rho_{ABE} = \rho_{AB} \otimes \sigma_E$, then the Schatten-p norm of this state can be written as $||\rho_{ABE}||_p = ||\rho_{AB}||_p||\sigma_E||_p$, as $||\sigma_E||_p = \mathrm{Tr}[\sigma_E^p]^{1/p}$ the only norm which cannot increase the quantum correlations applying any quantum channel on subsystem E is the Schatten-1. In Chap.2 we studied the properties of the trace distance, then the *1-norm geometrical discord* will keep these properties. As we discussed for a measure of quantumness of correlations to be acceptable it should satisfy some properties, the more important are the properties 76-79 of quantum discord [29].

Definition 86 (1-norm geometrical quantum discord)**.** *Given an bipartite quantum state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ we can define the 1-norm geometrical quantum discord:*

$$D_1(\rho_{AB}) = \inf_{\xi_{AB}\in\Omega_{QC}} ||\rho_{AB} - \xi_{AB}||_1, \tag{3.104}$$

where $\xi_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is a quantum-classical state.

The 1-norm geometrical quantum discord satisfy the following properties [7]:

Proposition 87 (ii)**.** *The 1-norm geometrical quantum discord is non-negative:*

$$D_1(\rho_{AB}) \geq 0. \tag{3.105}$$

Proposition 88 (iii)**.** *The 1-norm geometrical quantum discord vanishes if and only if the state is classically correlated.*

Proposition 89 (iv)**.** *The 1-norm geometrical quantum discord is invariant under local unitary operations:*

$$D_1(\rho_{AB}) = D_1((U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)). \tag{3.106}$$

Proof:(ii)-(iv). This properties come from the properties of the trace distance. For two density matrices $\rho, \sigma \in \mathcal{D}(\mathbb{C}^n)$ the trace distance respects:

(ii) Non-negative:

$$||\rho - \sigma||_1 \geq 0; \tag{3.107}$$

(iii) Vanishes if and only if the states are the same:

$$||\rho - \sigma||_1 = 0 \quad \Longleftrightarrow \quad \sigma = \rho; \tag{3.108}$$

(iv) Invariant under isometry operations:

$$||\rho - \sigma||_1 = ||V\rho V^\dagger - V\sigma V^\dagger||_1, \tag{3.109}$$

where $V \in \mathcal{U}(\mathbb{C}^n, \mathbb{C}^{n'})$ is an isometry.

[7] All of the properties described below came from the properties of the *trace distance* which were proved in last chapter.

□

As occurs with quantum discord, Proposition.79, the 1-norm geometrical quantum discord can increase by local operations on the measured system. The example for it is the same discussed above, which applying a local map that changes a classical-classical state to a classical-quantum state. The next property comes from the fact that trace distance is contractive under quantum channels. The proof of this statement is straightforward, because comes directly from the monotonicity of the trace distance.

Proposition 90 (v). *The 1-norm geometrical quantum discord is contractive under local maps on the non measured subsystem*

Proof. Consider a bipartite state ρ_{AB} and a local channel Φ_A, then the trace distance between ρ_{AB} and the closest quantum-classical state $\bar{\xi}_{AB}$ will satisfy:

$$||\rho_{AB} - \bar{\xi}_{AB}||_1 \geq ||\Phi_A \otimes \mathbb{I}_B(\rho_{AB}) - \Phi_A \otimes \mathbb{I}_B(\bar{\xi}_{AB})||_1, \tag{3.110}$$

the state $\Phi_A \otimes \mathbb{I}_B(\bar{\xi}_{AB})$ remains quantum-classical, although cannot be the closest state to $\Phi_A \otimes \mathbb{I}_B(\rho_{AB})$. If we named the closest as ξ^{Φ}_{AB}, hence $||\Phi_A \otimes \mathbb{I}_B(\rho_{AB}) - \Phi_A \otimes \mathbb{I}_B(\bar{\xi}_{AB})||_1 \geq ||\Phi_A \otimes \mathbb{I}_B(\rho_{AB}) - \xi^{\Phi}_{AB}||_1$, which implies:

$$||\rho_{AB} - \bar{\xi}_{AB}||_1 \geq ||\Phi_A \otimes \mathbb{I}_B(\rho_{AB}) - \xi^{\Phi}_{AB}||_1. \tag{3.111}$$

□

Another way to quantify quantum correlations is via the local disturbance under measurements [102]. In this way it is possible to define a local disturbance via 1-norm, it is named *negativity of quantumness* ($\mathcal{NQ}$) [113].

Definition 91. *Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, the negativity of quantumness is defined as the minimal amount of disturbance created in the quantum state, optimized over local projective measurements, measured via 1-norm distance:*

$$D_{\mathcal{NQ}}(\rho_{AB}) = \inf_{\Pi_B \in \mathcal{P}} ||\rho_{AB} - \mathbb{I}_A \otimes \Pi_B(\rho_{AB})||_1, \tag{3.112}$$

where $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$ is a local dephasing on the subsystem B.

In contrast to the local disturbance via Hilbert-Schmidt distance, the negativity of quantumness is not equal to the 1-norm geometrical discord for any bipartite system, they will be the same just for $2 \otimes 2$ systems [113]. However for measurements on both subsystems the negativity of quantumness is equal to the 1-norm geometrical discord for classical-classical states [113]:

Theorem 92. *Given a bipartite state ρ_{AB} the negativity of quantumness with measurement on both sides is:*

$$D_{\mathcal{NQ}}(\rho_{AB})^{\Pi_{AB}} = \inf_{\Pi_A \otimes \Pi_B \in \mathcal{P}} ||\rho_{AB} - \Pi_A \otimes \Pi_B(\rho_{AB})||_1, \tag{3.113}$$

and the 1-norm geometrical discord for classical-classical states is:

$$D_1(\rho_{AB})^{CC} = \inf_{\xi_{AB} \in \Omega_{CC}} ||\rho_{AB} - \xi_{AB})||_1, \tag{3.114}$$

they are the same measures of quantumness of correlations:

$$D_1(\rho_{AB})^{CC} = D_{\mathcal{NQ}}(\rho_{AB})^{\Pi_{AB}}. \tag{3.115}$$

The 1-norm geometrical discord and the negativity of quantumness can be calculated analytically for $2 \otimes 2$ systems [36,113,122,123]. They were also applied experimentally in a NMR device where the entropic properties of the quantum discord were inferred [121].

3.2.4 Relative entropy of quantumness and work deficit

The relative entropy is a measure of distance which does not satisfies the symmetric property to be a true distance [14], however for a given dephasing channel $\Pi \in \mathcal{P}(\mathbb{C}^n)$ acting on any state $\rho \in \mathcal{D}(\mathbb{C}^n)$ the support of the dephased state contains the support of the input state: $\text{supp}(\rho) \subseteq \text{supp}(\Pi[\rho])$, therefore the measure of quantumness of correlations based on the relative entropy remains finite for every composed state [14, 162].

Suppose Alice and Bob have a common composed system described by the state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ and they would like to extract work from this system, to perform a task. To attain their intention they can perform the *closed set of local operations and communicate classically* (CLOCC). This class of operations is composed by: *i)* addition of pure ancillas, *ii)* local unitary operations, *iii)* local dephasing channels. Sending states through a local dephasing channel represents the classical communication. If Alice and Bob are together in the same lab, they can extract work globally from the total system, then the total amount of information which Alice and Bob can extract from ρ_{AB} together is defined as the *total work* [117].

Definition 93. *The work which can be extracted from a quantum system, described by the state $\rho \in \mathcal{D}(\mathbb{C}^N)$, is defined as the change in the entropy:*

$$W_t(\rho) = \log_2 N - S(\rho), \tag{3.116}$$

$\log_2 N$ is the entropy of the maximally mixed state, for which it is not possible extract work and $S(\rho)$ is the von Neumann entropy of the state.

This function can be viewed as a function which measures information, such that if the state is a maximally mixed state no information can be extracted from it, therefore if the state is a pure state we have the maximum value of the information [79, 117]. The entropy function also represents the information about the system, although it represents the amount of information which we *can get to know* about the system and the function Eq.3.116 represents the amount of information we *already know*[8]. On the other hand, Alice and Bob cannot be in the same lab, therefore the information, or work, which can be extracted from the total state is restricted to be locally accessed. In the same way we defined the total information, it is possible to define a *local work*. Then Alice and Bob should apply CLOCC operation in order to obtain the maximal amount of local information:

$$W_l(\rho_{AB}) = \log_2 N - \sup_{\Gamma \in CLOCC} S(\Gamma[\rho_{AB}]), \tag{3.117}$$

where the state $\Gamma(\rho_{AB})$ is the state after the protocol. If after the protocol the whole state is with one part, for example A, then the dimension of the other will be 1 and its entropy will be zero, therefore the state in the end of the protocol will be $\Gamma(\rho_{AB}) = \rho_{AA'}$. When we are talking about local information, it does not mean getting some outputs via local measurements, it means applying a quantum channel, described by CLOCC operations, and taking a function of the state after the action of the channel, this function is named local information, or local work, and it is described by Eq.3.117 [79].

Actually the important function is not $W_l(\rho_{AB})$, the function which we are interested is the difference between the total work which can be extracted from the composed system described by ρ_{AB} and the work which we can extracted locally. This function is named *work deficit* and it measures the amount of work that is not possible to extract locally [117].

Definition 94. *Given a bipartite state ρ_{AB}, the information which two parts Alice and Bob cannot access, via CLOCC, is the work deficit:*

$$\Delta(\rho_{AB}) = W_t(\rho_{AB}) - W_l(\rho_{AB}). \tag{3.118}$$

[8] A pedagogic discussion about this definition, in the thermodynamic point of view, can be find in the Ref. [79].

From the definition of the total work and the local work we can rewrite the work deficit as:

$$\Delta(\rho_{AB}) = \inf_{\Gamma\in CLOCC} \{S(\Gamma[\rho_{AB}]) - S(\rho_{AB})\}. \tag{3.119}$$

Even though the total and the local work depend explicitly on the dimension of the system, the work deficit should not depend on the dimension of $\Gamma[\rho_{AB}]$. Adding local pure ancillas belongs to the CLOCC, it should not change the amount of work deficit. As we shall see below, the work deficit is a measure of correlations, then it must not change by the simple addition of a uncorrelated system [79, 112]. On the other hands this issue will be discussed better in Chap.5, where we shall study the effect of the embedding on the local measurement scenario.

In general the work deficit of a state ρ_{AB} is related with the relative entropy between the state ρ_{AB} and the final state $\Gamma[\rho_{AB}]$ [80].

Theorem 95. *The work deficit is upper bounded by the relative entropy of the state with the set of pseudo-classical correlated states $\mathcal{PC}$ [80]:*

$$\Delta(\rho_{AB}) \leq \inf_{\sigma_{AB}\in\mathcal{PC}} S(\rho_{AB}\|\sigma_{AB}), \tag{3.120}$$

where the set of pseudo-classically correlated states $\mathcal{PC}$ is the set of states which can be converted in classically correlated states via CLOCC operations without create any correlation. This set contain the set of classically correlated states: $\Omega_{CC} \subseteq \Omega_{CQ} \subseteq \mathcal{PC}$.

Proof. The final state $\Gamma[\rho_{AB}] \in \mathcal{PB}$. Then we can choose a basis $\mathcal{B}$, which is the basis that Alice and Bob applied the dephasing during the protocol, this basis is named *implementable product basis* (IPB). In the end of the protocol the final state has entropy:

$$S(\Gamma[\rho_{AB}]) = H(\rho_{AB}, \mathcal{B}), \tag{3.121}$$

where $H(\rho_{AB}, \mathcal{B}) = -\sum_k \langle\beta_k|\rho_{AB}|\beta_k\rangle \log_2 \langle\beta_k|\rho_{AB}|\beta_k\rangle$ for $\{|\beta_k\rangle\}_k \subset \mathcal{B}$. Then optimizing over all possible IPB, such that the state has the smaller entropy, we obtain:

$$\inf_{\Gamma\in CLOCC} S(\Gamma[\rho_{AB}]) \leq \inf_{\mathcal{B}} H(\rho_{AB}, \mathcal{B}). \tag{3.122}$$

We named the set $S_{\mathcal{B}}$ as the set of states which commute with the basis $\mathcal{B}$, then a state in this set we named $\rho_{\mathcal{B}}$. We also have the following relation to the Shannon entropy $H(\rho, \mathcal{B})$:

$$\begin{aligned}
\inf_{\sigma\in S_{\mathcal{B}}} \{S(\rho||\sigma) - S(\rho)\} &= \inf_{\sigma\in S_{\mathcal{B}}} [-\mathrm{Tr}(\rho\log_2\sigma)] && (3.123)\\
&= \inf_{\sigma\in S_{\mathcal{B}}} [-\mathrm{Tr}(\rho_{\mathcal{B}}\log_2\sigma)] + \mathrm{Tr}(\rho_{\mathcal{B}}\log_2\rho_{\mathcal{B}}) - \mathrm{Tr}(\rho_{\mathcal{B}}\log_2\rho_{\mathcal{B}}) && (3.124)\\
&= \inf_{\sigma\in S_{\mathcal{B}}} \{S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{B}}\|\sigma)\} && (3.125)\\
&= H(\rho, \mathcal{B}). && (3.126)
\end{aligned}$$

In Eq.3.124 we used the fact that $\mathcal{B}$ commutes with σ[9] and summed zero. In Eq.3.126 we used that $\sigma \in S_{\mathcal{B}}$, then $\inf_{\sigma\in S_{\mathcal{B}}} S(\rho_{\mathcal{B}}\|\sigma) = 0$ and $H(\rho, \mathcal{B}) = S(\rho_{\mathcal{B}})$. Now combining Eq.3.122 with Eq.3.126 we have:

$$\begin{aligned}
\Delta(\rho) &= \inf_{\Gamma\in CLOCC} S(\Gamma[\rho]) - S(\rho) && (3.127)\\
&\leq \inf_{\mathcal{B}\in IPB} H(\rho, \mathcal{B}) - S(\rho) && (3.128)\\
&= \inf_{\mathcal{B}\in\mathcal{PC}} S(\rho||\sigma). && (3.129)
\end{aligned}$$

In Eq.3.128 we substituted the Eq.3.122. In Eq.3.129 we substituted E.3.126 and rewrote

[9] This step will be calculated explicitly in Theorem 99.

$\inf_{\mathcal{B}\in IPB} \inf_{\sigma \in S_B} \{S(\rho||\sigma) = \inf_{\mathcal{B}\in \mathcal{PC}} S(\rho||\sigma)$ □

In the asymptotic limit (the limit of many copies) the work deficit quantifies the amount of pure states which can be extracted locally [51, 80]. However as resource cannot be created freely, the addition of pure local ancillas is not allowed, then it is replaced by the addition of maximally mixed states. The set of operations which contains: *i)* addition of maximal mixture states, *ii)* local unitary operations, *iii)* local dephasing channels, is named *noise local operations and classical communication* (NLOCC) [80]. The extraction of local pure states is a protocol, whose the goal is to extract resource, where the set of available operations is the NLOCC operations and the set of free resource states is composed just by the maximal mixture state, which is the only state which does not have any local purity [82]. It remains an open question if the CLOCC class and the NLOCC class are equivalent [79].

In the limit of one copy, the work deficit can quantify quantum correlations present in a given composed system [116]. The scenario where Alice and Bob can perform many steps of classical communication is named *two way*, and the work deficit is named *two-way work deficit*. In this case they can perform measurements and communicate in each step of the protocol. Mathematically the two-way work deficit does not have a closed expression [79]. As discussed above, we can "active" quantum correlations performing operations on the measured system. Therefore this many steps scenario is not good to quantify quantum correlations, because if Alice and Bob can perform dephasing channels that would not commute with the dephasing applied in the before, then the only state which remains invariant, under the actions of the arbitrary dephasing channels, is the maximally mixed state. In this way, it is necessary a one round description, where the only thing that Alice and Bob can do is communicate in the end of the protocol. Following this idea, it is possible to define two classes of work deficit, the *one way work deficit*, which just one side can communicate. If Bob communicates to Alice, the state created in the end of the protocol is a quantum-classical state (or a classical-quantum state if Alice communicates in the end of the protocol).

Definition 96 (one way work deficit)**.** *Given a bipartite state ρ_{AB}, the work deficit with just one side communication is named one way work deficit [117]:*

$$\Delta^{\rightarrow}(\rho_{AB}) = \min_{\Pi_B \in \mathcal{P}} \{S(\mathbb{I}_A \otimes \Pi_B[\rho_{AB}]) - S(\rho_{AB})\}, \tag{3.130}$$

where $\Pi_B \in \mathcal{P}(\mathbb{C}_B)$ is a local dephasing on subsystem B. We shall write $\Delta^{\rightarrow}(\rho_{AB})$ when the communication is from A to B and $\Delta^{\leftarrow}(\rho_{AB})$ otherwise.

Another definition for the work deficit is defined when both Alice and Bob communicate in the end of the protocol, this is named *zero way work deficit*. The state created in the end of the protocol is a classical-classical state.

Definition 97 (zero way work deficit)**.** *Given a bipartite state ρ_{AB}, the work deficit with no communication until the end of the protocol is named zero work deficit [117]:*

$$\Delta^{\emptyset}(\rho_{AB}) = \min_{\Pi_A \otimes \Pi_B \in \mathcal{P}} \{S(\Pi_A \otimes \Pi_B[\rho_{AB}]) - S(\rho_{AB})\}, \tag{3.131}$$

where $\Pi_A \otimes \Pi_B \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is a local dephasing on subsystem A and B.

K. Modi *et. al* proposed a measure of quantumness of correlation defined as the relative entropy of the state and the set of classical correlated states [112]. This measure is named *relative entropy of quantumness*.

Definition 98 (Relative entropy of quantumness)**.** *The relative entropy of quantumness $D(\rho_{AB})_{QC}$ for a given state ρ_{AB} is defined as the minimum relative entropy over the set of quantum-classical states [112]:*

$$D(\rho_{AB})_{QC} = \min_{\xi_{AB} \in \Omega_{QC}} S(\rho_{AB} \| \xi_{AB}), \tag{3.132}$$

where Ω_{QC} is the set of quantum-classical states.

The relative entropy of quantumness for classical-classical states is denoted as $D(\rho_{AB})_{CC}$. It is analogous to Eq.3.133 when the optimization is taken over the set of classical-classical states Ω_{CC}:

$$D(\rho_{AB})_{CC} = \min_{\xi_{AB}\in\Omega_{CC}} S(\rho_{AB}\|\xi_{AB}). \tag{3.133}$$

As discussed in the limit of one copy the one way and the zero way work deficit are quantifiers of quantum of correlations. For these cases the equality is attained in the last theorem.

Theorem 99. *The 1-way work deficit is equal to the relative entropy of quantumness for quantum-classical states [79, 112]:*

$$D(\rho_{AB})_{QC} = \Delta^{\rightarrow}(\rho_{AB}), \tag{3.134}$$

Proof. Suppose $\xi_\rho \in \Omega_{QC}$ is the state which optimizes the Eq.3.133 and another state $X = \sum_k (\mathbb{I}_A \otimes |k\rangle\langle k|)\rho_{AB}(\mathbb{I}_A \otimes |k\rangle\langle k|)$, such that $[\xi_\rho, X] = 0$. As the state ξ_ρ is the nearest state, measuring via the relative entropy, the next expression must hold:

$$S(\rho_{AB}\|X) - S(\rho_{AB}\|\xi_{AB}) \geq 0. \tag{3.135}$$

Then calculating explicitly the relative entropies:

$$\begin{aligned} S(\rho\|X) &= -S(\rho) - \mathrm{Tr}[\rho \log X] &(3.136)\\ &= -S(\rho) - \sum_k \mathrm{Tr}[(\mathbb{I}_A \otimes |k\rangle\langle k|)\rho_{AB} \log X] &(3.137)\\ &= -S(\rho) - \sum_k \mathrm{Tr}[(\mathbb{I}_A \otimes |k\rangle\langle k|)\rho_{AB}(\mathbb{I}_A \otimes |k\rangle\langle k|) \log X] &(3.138)\\ &= -S(\rho) + S(X), &(3.139)\end{aligned}$$

we used the idempotent property of projectors, the cyclic property of the trace and the fact that $[\xi_\rho, X] = 0$, respectively. Repeating the same algebra for $S(\rho_{AB}\|\xi_{AB})$:

$$\begin{aligned} S(\rho\|\xi_\rho) &= -S(\rho) - \mathrm{Tr}[\rho \log \xi_\rho] &(3.140)\\ &= -S(\rho) - \mathrm{Tr}[X \log \xi_\rho]. &(3.141)\end{aligned}$$

Therefore:

$$S(\rho_{AB}\|X) - S(\rho_{AB}\|\xi_{AB}) = S(X) + \mathrm{Tr}[X \log \xi_\rho] = -S(X\|\xi_\rho) \leq 0, \tag{3.142}$$

which implies that $S(X\|\xi_\rho)$, hence $X = \xi_\rho$. □

The same is valid for the zero way work deficit and the relative entropy of quantumness for classical-classical states:

$$D(\rho_{AB})_{CC} = \Delta^{\emptyset}(\rho_{AB}). \tag{3.143}$$

The one way and zero way work deficits quantify quantum correlations beyond the quantum entanglement, therefore we should be able to compare these two classes of quantum correlations. For the relative entropy this comparison is natural, and it comes directly from the relative entropy of the fact that CLOCC is a sub class of LOCC operations.

Proposition 100. *The work deficit is lower bounded by the relative entropy of entanglement:*

$$\Delta(\rho) \geq E_r(\rho). \tag{3.144}$$

Proof. As CLOCC $\subset$ LOCC the minimization of the relative entropy over these sets:

$$\inf_{\Gamma\in CLOCC} S(\rho||\Gamma[\rho]) \geq \inf_{\Gamma\in LOCC} S(\rho||\Lambda[\rho]) = \inf_{\sigma\in Sep} S(\rho||\sigma), \tag{3.145}$$

where *Sep* is the set of separable states. Given the definition of the relative entropy of entanglement and the work deficit:

$$\Delta(\rho) = \inf_{\Gamma \in CLOCC} S(\rho||\Gamma[\rho]) \geq \inf_{\sigma \in Sep} S(\rho||\sigma) = E_r(\rho). \tag{3.146}$$

□

The separable states are the states which can be created via LOCC operations, a given bipartite separable state σ_{AB}^{sep} can be written as a convex combination of pure product states:

$$\sigma_{AB}^{sep} = \sum_k q_k \left|\phi_k\right\rangle\left\langle\phi_k\right|_A \otimes \left|\psi_k\right\rangle\left\langle\psi_k\right|)_B, \tag{3.147}$$

as discussed this state is classically correlated if the states satisfy $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$ or/and $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$. In other words a classically correlated state is a convex combination of pure product states which form a orthogonal basis. Therefore a separable pure state is also classically correlated, then it is natural to expect that a measure of quantumness of correlations captures this characteristic. Indeed for bipartite pure states the work deficit quantifies the quantum correlations in the pure state, which is just the entanglement.

Proposition 101. *For bipartite pure states $|\psi\rangle_{AB} \in \mathbb{C}_A \otimes \mathbb{C}_B$, the work deficit is equal to the relative entropy of entanglement [117]:*

$$\Delta(\Psi_{AB}) = E_r(\Psi_{AB}) = S(\rho_A), \tag{3.148}$$

where $\Psi_{AB} = |\psi\rangle\langle\psi|_{AB}$.

Proof. It is well known that for pure states the relative entropy of entanglement is equal to the von Neumann entropy of the marginal. As the work deficit is lower bounded by the relative entropy of entanglement, if we find a local dephasing channel, such that the work deficit is equal to the relative entropy of entanglement, it is the optimal protocol which minimize the work deficit, once that:

$$S(\Psi||\Pi_A \otimes \Pi_B[\Psi]) \geq \Delta(\Psi) \geq E_r(\Psi), \tag{3.149}$$

where $\Pi_A \otimes \Pi_B \in \mathcal{P}(\mathbb{C}_A \otimes \mathbb{C}_B)$. Given the Schmidt decomposition of the state $|\psi\rangle_{AB} = \sum_i c_i |a_i\rangle |b_i\rangle$, hence if we apply a dephasing channel on the Schmidt basis:

$$\Pi_A \otimes \Pi_B[\Psi] = \sum_{ij} c_i c_j \Pi_A[|a_i\rangle\langle a_j|] \otimes \Pi_B[|b_i\rangle\langle b_j|] = \sum_i c_i^2 |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|. \tag{3.150}$$

Thus the von Neumann entropy of $\Pi_A \otimes \Pi_B[\Psi]$ is equal to the von Neumann entropy of the marginals of Ψ:

$$S(\Pi_A \otimes \Pi_B[\Psi]) = -\sum_i (c_i^2) \log_2 c_i^2 = S(\rho_A). \tag{3.151}$$

□

An interesting corollary of this proposition is that the quantum discord is equal to the work deficit for pure states, because it is also equal to the entropy of entanglement for pure states, as calculated in Proposition 83.

Chapter 4

Interplay between quantum entanglement and quantumness of correlations.

In this chapter we shall explain three different ways to relate quantum entanglement and quantumness of correlations. In the first two sections we present two ways to relate entanglement and quantumness of correlations. In Sec.4.1, we revise the literature about the expressions calculated via the well known Koashi-Winter relation. Section 4.2 is intended to the description of the quantum correlation activation protocol. We have a special interest in this protocol because we shall use it on Chap.6 to characterize the classical correlated states in indistinguishable particle systems. In Sec.4.3, we present an original work which relates the quantumness of correlations on the geometrical approach with the witnessed entanglement.

4.1 Monogamy relation: entanglement, classical correlations and quantumness of correlations

Given a bipartite system $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, then we can purify this state in a larger space $\mathbb{C}_{ABE}$, of the dimension: $\dim(\mathbb{C}_{ABE}) = \dim(A) \cdot \dim(B) \cdot \text{rank}(\rho_{AB})$. The purification process will create quantum correlations between the system AB and the purification system E, unless the state is already pure. Intrinsically to the process of purification there is a restriction in the amount of correlations which the state can share with the purification system, otherwise the amount of classical correlations would be free. This balance between the correlations for tripartite states can be understood via the Koashi-Winter relation.

Given the definition of the classical correlations for a bipartite state ρ_{AB}:

$$J(A:B)_{\rho_{AB}} = \max_{\mathbb{I}\otimes \in \mathcal{P}} I(A:X)_{\mathbb{I}\otimes(\rho_{AB})}, \tag{4.1}$$

where $I(A:X)_{\mathbb{I}\otimes(\rho_{AB})}$ is the mutual information of the post-measured state $\mathbb{I} \otimes (\rho_{AB})$, and the optimization is taken over all local POVM measurement maps $\in \mathcal{P}(\mathbb{C}_B, BC_X)$.

Given also the definition of the entanglement of formation of a bipartite state ρ_{AB}:

$$E_f(\rho_{AB}) = \min_{\xi_\rho = \{p_i, |\psi_i\rangle\langle\psi_i|\}_i} \sum_i p_i E(|\psi_i\rangle), \tag{4.2}$$

where the optimization is taken over all possible convex hull defined by the ensemble $\xi = \{p_i, |\psi_i\rangle\langle\psi_i|\}_i$, such that $\rho_{AB} = \sum_i p_i, |\psi_i\rangle\langle\psi_i|$, and $E(|\psi_i\rangle)$ is the entropy of entanglement of $|\psi_i\rangle$.

Theorem 102 (Koashi-Winter relation [89]). *Considering $\rho_{ABE} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_E)$ a pure state then:*

$$J(A:E)_{\rho_{AE}} = S(\rho_A) - E_f(\rho_{AB}), \tag{4.3}$$

where $\rho_X = Tr_Y[\rho_{YX}]$.

Proof. Suppose $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is written in the convex combination which optimizes the entanglement of formation $E_f(\rho_{AB}) = \sum_i p_i S(Tr_B[|\psi_i\rangle\langle\psi_i|])$. To obtain the classical correlations in system AE we should relate this decomposition with a measurement on the subsystem E. There exists a measurement $\{M_j^E\}$ on system E such that $\rho'_{ABE} = \sum_j Tr_E[\rho_{ABE}(\mathbb{I}_{AB} \otimes M_j^E)] \otimes |e_j\rangle\langle e_j|_E$ and $Tr_E[\rho'_{ABE}] = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. As we are calculating the classical correlations on ρ_{AE}, the post-measurement state will be:

$$\rho'_{AE} = \sum_j p_j Tr_B[|\psi_j\rangle\langle\psi_j|] \otimes |e_j\rangle\langle e_j|, \tag{4.4}$$

in this way we can compute the mutual information of the post-measurement state:

$$\begin{aligned} I(A:E)_{\rho'_{AE}} &= S(\rho_A) + S(\rho'_E) - S(\rho'_{AE}), & (4.5)\\ &= S(\rho_A) + H(E) - H(E) - \sum_i p_i S(Tr_A[|\psi_j\rangle\langle\psi_j|]), & (4.6)\\ &= S(\rho_A) - \sum_i p_i S(Tr_A[|\psi_j\rangle\langle\psi_j|]), & (4.7)\\ &= S(\rho_A) - E_f(\rho_{AB}), & (4.8) \end{aligned}$$

we used the property of the Shannon entropy for a block diagonal state, where $Tr_B[|\psi_j\rangle\langle\psi_j|] = Tr_A[|\psi_j\rangle\langle\psi_j|]$ and $E_f(\rho_{AB}) = \sum_i p_i S(Tr_B[|\psi_i\rangle\langle\psi_i|])$. As by definition $J(A:E)_{\rho_{AE}} \geq I(A:E)_{\rho'_{AE}}$, then

$$J(A:E)_{\rho_{AE}} \geq S(\rho_A) - E_f(\rho_{AB}). \tag{4.9}$$

Now we shall prove equality conversely. Given ρ_{AE}, there exists a POVM $\mathcal{A} \in \mathcal{P}(\mathbb{C}_E, \mathbb{C}_{E'})$ with rank-1 elements $\{A_l\}$, such that $Tr_E[A_l \rho_{AE}] = q_l \rho_l^A$ and it optimizes the classical correlations $J(\rho_{AE}) = S(\rho_A) - \sum_l q_l S(\rho_l^A)$. As the elements of the POVM are rank-1, $A_l = |\mu_l\rangle\langle\mu_l|$, and the state ρ_{ABE} is pure, the state after local measurement on E will be described by an ensemble of pure states:

$$\rho'_{ABE} = \sum_l Tr_E[\rho_{ABE}(\mathbb{I}_{AB} \otimes |\mu_l\rangle\langle\mu_l|)] \otimes |e_l\rangle\langle e_l| = \sum q_l |\phi_l\rangle\langle\phi_l| \otimes |e_l\rangle\langle e_l|. \tag{4.10}$$

It is easy to understand once that $\rho_{ABE} = |\kappa\rangle\langle\kappa|$, and the pure state can be written in the bipartite Schmidt decomposition $|\kappa\rangle = \sum_n c_n |n\rangle_{AB} \otimes |n\rangle_E$, if $\langle\mu_l|n\rangle = r_{ln}$, therefore

$$Tr_E[\rho_{ABE}(\mathbb{I}_{AB} \otimes |\mu_l\rangle\langle\mu_l|)] = \sum_{ij} c_i r_{li} c_j r^*_{lj} |i\rangle\langle j|_{AB} = \left(\sum_i c_i r_{li} |i\rangle_{AB}\right)\left(\sum_j c_j r^*_{lj} \langle j|_{AB}\right) = q_l |\phi_l\rangle\langle\phi_l|. \tag{4.11}$$

Calculating the mutual information of $\rho'_{AE} = Tr_B[\rho'_{ABE}]$:

$$I(A:E)_{\rho'_{AE}} = S(\rho_A) - \sum_l q_l S(Tr_B[|\phi_l\rangle\langle\phi_l|]), \tag{4.12}$$

as the POVM $\mathcal{A}$ is the optimal measurement in the calculation of the classical correlations it implies $I(A:E)_{\rho'_{AE}} = J(A:E)_{\rho_{AE}}$. By the definition of the entanglement of formation: $E_f(\rho_{AB}) \leq \sum_l q_l S(Tr_B[|\phi_l\rangle\langle\phi_l|])$ for any decomposition $\{p_l, |\phi_l\rangle\langle\phi_l|\}$. Substituting the mutual information on Eq.4.55:

$$J(A:E)_{\rho_{AE}} \leq S(\rho_A) - E_f(\rho_{AB}). \tag{4.13}$$

Given Eq.4.9 and Eq.4.13 it proves the theorem. □

The Koashi-Winter equation quantifies the amount of entanglement among A and B, considering that the former is classically correlated with another system C. This property is interesting once that it is related with the monogamy of entanglement [37], where the amount of entanglement shared by three parts is limited, and this limitation is given by the amount of classical correlations among the parties and the entropy of them (related with the rank of the density matrices). This property is valid for any tripartite state as stated in the following corollary:

Corolary 103. *For any tripartite state $\rho_{ABC} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_C)$, it follows:*

$$E_f(\rho_{AB}) + J(A:C)_{\rho_{AC}} \leq S(\rho_A). \tag{4.14}$$

The equality holds for ρ_{ABC} pure.

Proof. If ρ_{ABC} is not a pure state, there exists a purification ρ_{ABCE}, such that we can separate the composed Hilbert space in three $\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_{CE}$, then follows the last theorem:

$$J(A:CE)_{\rho_{ACE}} + E_f(\rho_{AB}) = S(\rho_A), \tag{4.15}$$

therefore as the classical correlations are monotonic under local maps, then taking the trace over the system E we have $J(A:CE)_{\rho_{ACE}} \geq J(A:C)_{\rho_{AC}}$. □

As the Shannon entropy of ρ_A represents the effective size of A in qubits [143], this size can be viewed as the capacity of the system A makes correlations with other systems B and C [89]. In other words, this means that the existence of the quantum or classical correlations between A and another system B is enough to restrict the amount of quantum or classical correlations which A can make with another system C.

As the quantumness of correlations present in a composed system can be viewed as the difference between the total correlations and the classical correlations, then for a pure tripartite state $\rho_{ABE} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B \otimes \mathbb{C}_E)$, we can sum the mutual information $I(A:E)_{\rho_{AE}}$ on both sides of the Koashi-Winter relation, Eq.4.3, and obtain a monogamy expression for the entanglement of formation of the state ρ_{AB} in function of the quantum discord [60]:

$$D(A:E)_{\rho_{AE}} = E_f(\rho_{AB}) - S(A|E)_{\rho_{AE}}, \tag{4.16}$$

where $D(A:E)_{\rho_{AE}}$ is the quantum discord of the state ρ_{AE} with local measurement on the subsystem E and $S(A|E)_{\rho_{AE}} = S(AE) - S(E)$ is the conditional entropy. As the label in the states is arbitrary we can rewrite this expression changing the labels $E \to B$ and vice versa to obtain $D(A:B)_{\rho_{AB}} = S(A|B)_{\rho_{AB}} - E_f(\rho_{AE})$, taking the sum between this and Eq.4.16:

$$D(A:E)_{\rho_{AE}} + D(A:B)_{\rho_{AB}} = E_f(\rho_{AE}) + E_f(\rho_{AB}), \tag{4.17}$$

as the total state is pure $S(A|E)_{\rho_{AE}} = -S(A|B)_{\rho_{AB}}$. This expression means that the total amount of entanglement which one part can have with two different other parts is restricted by the sum of the amount of quantum correlations which this part can have. This restriction is also valid in the opposite side [60].

From Eq.4.16 it is possible to calculate an interesting expression which relates the irreversibility of the entanglement distillation protocol and quantum discord [38]. This is irreversible because, as we discussed in Chapter.3, the entanglement cost can be larger than the distillable entanglement, in other words we need more maximally entangled states to create such a state than we can extract. The entanglement cost can be defined as the regularization of the entanglement of formation [75]:

Definition 104. *For a mixed state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ the regularization of the entanglement of formation $E_f(\rho_{AB})$ results in the entanglement cost:*

$$E_C(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} E_f(\rho_{AB}^{\otimes n}). \tag{4.18}$$

The Hashing inequality says that the distillable entanglement of ρ_{AB} is lower bounded by the coherent information $I(A\rangle B)_{\rho_{AB}} = -S(A|B)$ [54]. As the coherent information can increase under LOCC it is possible to optimize it under LOCC attaining the distillable entanglement [54].

Definition 105. *The regularized coherent information after optimization over LOCC for a mixed state ρ_{AB} gives the distillable entanglement:*

$$E_D(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} I(A\rangle B)_{(V_n\otimes\mathbb{I})\rho_{AB}^{\otimes n}}, \tag{4.19}$$

where $V_n \otimes \mathbb{I}$ acts locally on the n copies of ρ_{AB}.

It is also possible to define the regularized quantum discord:

Definition 106. *The regularized quantum discord can be defined as the quantum discord of a state ρ_{AB} in the limit of many copies:*

$$D^\infty(A:B)_{\rho_{AB}} = \lim_{n\to\infty} \frac{1}{n} D(A:B)_{\rho_{AB}^{\otimes n}}. \tag{4.20}$$

Therefore similarly to Eq.4.16 in the limit of many copies:

$$D(A:E)_{\rho_{AE}^{\otimes n}} = E_f(\rho_{AB}^{\otimes n}) - S(A|E)_{\rho_{AE}^{\otimes n}}, \tag{4.21}$$

taking the regularization we have:

$$D^\infty(A:E)_{\rho_{AE}} = E_C(\rho_{AB}) - S(A|E)_{\rho_{AE}}, \tag{4.22}$$

as the conditional entropy is additive $S(A|E)_{\rho_{AE}^{\otimes n}} = nS(A|E)_{\rho_{AE}}$. Then for states which satisfy the following theorem:

Theorem 107 (M. Cornelio *et al.* [38])**.** *For every mixed entangled state ρ_{AB}, if*

$$E_D(\rho_{AB}) = \frac{1}{n} I(A\rangle B)_{(V_n\otimes\mathbb{I})\rho_{AB}^{\otimes n}} \tag{4.23}$$

$$E_C(\rho_{AB}) = \frac{1}{k} E_F(\rho_{AB}^{\otimes n}), \tag{4.24}$$

for a finite number of n and k, the entanglement is irreversible $E_C(\rho_{AB}) > E_D(\rho_{AB})$.

Taking the limit of many copies, the equation can be rewritten as:

$$D^\infty(A:E)_{\sigma_{AE}} = E_C(\sigma_{AB}) - E_D(\sigma_{AB}), \tag{4.25}$$

where $\sigma_{AB} = (V_k \otimes \mathbb{I})\rho_{AB}$ and $E_D(\sigma_{AB}) = kE_D(\rho_{AB})$. The quantum discord $D^\infty(A:E)_{\sigma_{AE}}$ in this context can be viewed as the minimal amount of entanglement lost in the distillation protocol, for states belonging to the class described in the theorem [38]. This expression can be viewed as an operational interpretation for quantum discord, where the quantum discord between the system and the purification system restricts the amount of e-bits lost in the distillation process. A similar interpretation can be obtained via the state merging protocol [83], as we discussed in Chap.3, Alice (A), Bob (B) and the Environment (E) share a pure tripartite state ρ_{ABE}, she would like to send her state to Bob, keeping the coherence with the system E. They can perform this protocol consuming an amount of entanglement in the process, the amount of entanglement is the regularized quantum discord $D^\infty(A:E)_{\rho_{AE}}$ [32,106].

In addition to the above relations, some upper and lower bounds between quantum discord and entanglement of formation have been calculated via the Koashi-Winter relation and the properties of entropy [166–169]. The Eq.4.16 was also used to calculate the quantum discord and the entanglement of formation analytically for systems with rank-2 and dimension $2\otimes n$ [33,59,92].

4.2 Activation protocol

Since the beginning we are describing the measurement process as a classical statistical inference obtained via a dephasing channel on the state. Physically we can describe it as an interaction between the measurement apparatus and the system, followed by a projective measurement on the apparatus. Suppose we have a state $\rho_{\mathcal{S}} = \sum_k \lambda_k |k\rangle\langle k| \in \mathcal{D}(\mathbb{C}_{\mathcal{S}})$, the input state which take in account the state of the system and the state of the measurement apparatus can be described as $\rho_{\mathcal{S}:\mathcal{M}} = \rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{M}}$. The interaction between the system and the apparatus ancillary state will be done via a unitary evolution: $U_{\mathcal{S}:\mathcal{M}} \in \mathcal{U}(\mathbb{C}_{\mathcal{S}} \otimes \mathbb{C}_{\mathcal{M}})$, such that $Tr_{\mathcal{M}}[U_{\mathcal{S}:\mathcal{M}}\rho_{\mathcal{S}:\mathcal{M}}U^{\dagger}_{\mathcal{S}:\mathcal{M}}] = \sum_l \Pi_l \rho_{\mathcal{S}} \Pi_l^{\dagger}$. A unitary operation which satisfies this condition is given by:

$$U_{\mathcal{S}:\mathcal{M}} |k\rangle_{\mathcal{S}} |0\rangle_{\mathcal{M}} = |k\rangle_{\mathcal{S}} |k\rangle_{\mathcal{M}}, \tag{4.26}$$

where $\{|k\rangle\}$ is an orthonormal basis in $\mathbb{C}_{\mathcal{S}}$. If the orthogonal basis $\{|k\rangle\langle k|\}$ is the canonical basis, this interaction is just the Cnot gate [114]. Therefore, after the interaction, the state will be:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}:\mathcal{M}})U^{\dagger}_{\mathcal{S}:\mathcal{M}} = \sum_k \lambda_k |k\rangle\langle k|_{\mathcal{S}} \otimes |k\rangle\langle k|_{\mathcal{M}}. \tag{4.27}$$

The interaction between the system and the measurement apparatus results in a classical correlated state between the system and the apparatus, hence applying a projective measurement on the apparatus state we can recovery the state of the system.

Suppose now the state of the system is a composed system, for example a bipartite system[1] $\mathbb{C}_{\mathcal{S}} = \mathbb{C}_A \otimes \mathbb{C}_B$, and measurement will be performed locally in each system, then $\mathbb{C}_{\mathcal{M}} = \mathbb{C}_{\mathcal{M}_A} \otimes \mathbb{C}_{\mathcal{M}_B}$. The unitary which represents the interaction between the system and the measurement apparatus is $U_{\mathcal{S}:\mathcal{M}} = U_{A:\mathcal{M}_A} \otimes U_{B:\mathcal{M}_B}$, and the post-measured state is:

$$\tilde{\rho}_{\mathcal{S}} = Tr_{\mathcal{M}}[U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|)U_{\mathcal{S}} : \mathcal{M}^{\dagger}] = \sum_{k,l} \Pi_k^A \otimes \Pi_l^B \rho_{AB} \Pi_k^{\dagger A} \otimes \Pi_l^{\dagger B}. \tag{4.28}$$

As described above, the measurement process consists in interacting the system with an ancilla, which represents the measurement apparatus, and then applying a projective measurement on the state of the apparatus. Although the dimension of the ancilla is arbitrary, we can choose to couple an ancilla with the same size of the state, such that the measurement represents an general POVM measurement. To obtain it we can write an input state $\rho_{\mathcal{S}':\mathcal{M}} = \rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}} \otimes |0\rangle\langle 0|_{\mathcal{M}}$, where $|0\rangle\langle 0|_{\mathcal{E}}$ is an ancillary state on space $\mathbb{C}_{\mathcal{E}}$, then the interaction with the apparatus will be given by a unitary evolution $U_{\mathcal{S}':\mathcal{M}}$ such that the post-measured state is:

$$\tilde{\rho}_{\mathcal{S}} = Tr_{\mathcal{M}}[U_{\mathcal{S}':\mathcal{M}}\rho_{\mathcal{S}':\mathcal{M}}U^{\dagger}_{\mathcal{S}':\mathcal{M}}] = \sum_l \Pi_l(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}})\Pi_l, \tag{4.29}$$

as by the Naimark's theorem $Tr[\Pi_l(\rho_{\mathcal{S}} \otimes |0\rangle\langle 0|_{\mathcal{E}})] = Tr[E_l \rho_{\mathcal{S}}]$ for $E_l = (\mathbb{I} \otimes \langle 0|)\Pi_l(\mathbb{I} \otimes |0\rangle)$ it represents a general measurement.

A general bipartite state can be written as $\rho = \sum_{i,j} |i\rangle\langle j| \otimes O_{i,j}$, where $O_{i,j}$ is an Hermitean operator with trace different from zero. Then if the measurement is performed only on the subsystem A, the state $\tilde{\rho}_{\mathcal{S}:\mathcal{M}}$ after the interaction with the measurement apparatus will be:

$$\begin{aligned} \tilde{\rho}_{\mathcal{S}:\mathcal{M}} &= U_{\mathcal{S}:\mathcal{M}}(\rho_{\mathcal{S}:\mathcal{M}})U^{\dagger}_{\mathcal{S}:\mathcal{M}} && (4.30)\\ &= U_{A:\mathcal{M}_A} \otimes \mathbb{I}_B \left(\sum_{i,j} |i\rangle\langle j|_A \otimes |0\rangle\langle 0|_{\mathcal{M}_A} \otimes O^B_{i,j} \right) U^{\dagger}_{A:\mathcal{M}_A} \otimes \mathbb{I}_B && (4.31)\\ &= \sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_{\mathcal{M}_A} \otimes O^B_{i,j}, && (4.32) \end{aligned}$$

[1] Although the description made here is also valid for multipartite systems, as has been done by the authors in one of the seminal papers [129].

where differently from the case which the measurement is applied on the whole system, for local measurements the system and the measurement apparatus can be quantum correlated after the interaction between them. The way how the interaction can create quantum correlations and what kind of correlations can be created for a given state are managed by the following theorem:

Theorem 108 ([129,150]). *A state is classical correlated (has no quantumness of correlations) if and only if there exists a unitary operation such that the post interaction state is separable with respect to system and measurement apparatus.*

Proof. Here we will prove the general case, which we can apply the measurement on both systems.

If : If the state is classical correlated:

$$\rho_{\mathcal{S}} = \sum_{k,j} p_{k,j} \left|a_k, b_j\right\rangle\left\langle a_k, b_j\right|_{\mathcal{S}}, \tag{4.33}$$

the state after the interaction with the measurement apparatus represented by the unitary operation $U_{A:\mathcal{M}_A} \otimes U_{B:\mathcal{M}_B}$ will be:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = \sum_{k,j} p_{k,j} \left|a_k, b_j\right\rangle\left\langle a_k, b_j\right|_{\mathcal{S}} \otimes \left|a_k, b_j\right\rangle\left\langle a_k, b_j\right|_{\mathcal{M}}, \tag{4.34}$$

which is clearly separable.

Only if : Given a general separable state between the system and the measurement apparatus:

$$\tilde{\rho}_{\mathcal{S}:\mathcal{M}} = \sum_{\alpha} p_\alpha \left|\phi_\alpha\right\rangle\left\langle\phi_\alpha\right|_{\mathcal{S}} \otimes \left|\psi_\alpha\right\rangle\left\langle\psi_\alpha\right|_{\mathcal{M}}, \tag{4.35}$$

as the interaction is unitary, there is a convex combination such that $\rho_{\mathcal{S}} = \sum_\alpha p_\alpha \left|\kappa_\alpha\right\rangle\left\langle\kappa_\alpha\right|$, therefore the interaction must act in the following way:

$$U_{\mathcal{S}:\mathcal{M}} \left|\kappa_\alpha\right\rangle \left|0\right\rangle = \left|\phi_\alpha\right\rangle \left|\psi_\alpha\right\rangle. \tag{4.36}$$

On the other hand, as the state $\rho_{\mathcal{S}}$ is bipartite, the pure states $\{\left|\kappa_\alpha\right\rangle\}$ can be written in general as: $\left|\kappa_\alpha\right\rangle = \sum_{l,i} c^\alpha_{l,i} \left|a^\alpha_l\right\rangle \left|b^\alpha_i\right\rangle$, after the interaction the state is:

$$U_{\mathcal{S}:\mathcal{M}} \left|\kappa_\alpha\right\rangle \left|0\right\rangle = \sum_{l,j} c^\alpha_{l,j} \left|a^\alpha_l, b^\alpha_j\right\rangle_{\mathcal{S}} \otimes \left|a^\alpha_l, b^\alpha_j\right\rangle_{\mathcal{M}}. \tag{4.37}$$

As the state in Eq.4.37 must be separable, it implies that the coefficients must satisfy:

$$c^\alpha_{i,j} = c_{f(\alpha)}\delta_{i,j;f(\alpha)} \quad \text{and} \quad \left|c_{f(\alpha)}\right| = 1 \tag{4.38}$$

where $f(\alpha) \in \mathbb{N}^2$. As $f(\alpha)$ are orthogonals it proves the theorem.

□

Therefore the local measurement process will create entanglement between the system and the measurement apparatus, for any unitary interaction, when the state of the system has quantum correlations. The unitary evolution depends to the basis of the ancillary pure state, which is coupled on the system before the measurement. Then we can fix the basis of the ancilla and change the basis of the system, in other words, we can rewrite the unitary evolution as $U_{\mathcal{S}:\mathcal{M}} = C_{\mathcal{S}:\mathcal{M}}(U_{\mathcal{S}} \otimes \mathbb{I}_{\mathcal{M}})$, where for bipartite systems $U_{\mathcal{M}} = U_A \otimes U_B$ is a local unitary operation and $C_{\mathcal{S}:\mathcal{M}} = C_{A:\mathcal{M}_A} \otimes C_{B:\mathcal{M}_B}$ is the system-apparatus Cnot gate, with the system as the control and the apparatus as the target. It is possible to quantify the amount of quantum correlation in a given system via the amount of entanglement which it can have with the measurement apparatus.

Definition 109. *Each measure of entanglement used to quantify the entanglement between the system and the apparatus will result in a measure of quantumness of correlations.*

$$Q_E(\rho_{\mathcal{S}}) = \min_{U_{\mathcal{S}}} E_Q(\rho_{\mathcal{S}:\mathcal{M}}). \tag{4.39}$$

Different entanglement measures will lead, in principle, to different quantifiers for the quantumness of correlations. The only requirement is that the entanglement measure be monotone under LOCC maps [128, 129, 150]. Other measures of quantumness can be recovered with the activation protocol: the quantum discord [150], one-way work deficit [150], relative entropy of quantumness [129] and the geometrical measure of discord via trace norm [113], are some examples.

From the Eq.4.39 it is straightforward to calculate an interplay between entanglement and its related measure of quantumness of correlation.

Proposition 110 (M. Piani and G. Adesso [113])**.** *For $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$:*

$$Q_E(\rho_{AB}) \geq E_Q(\rho_{AB}), \tag{4.40}$$

where Q_E and E_Q are related via Eq.4.39.

Proof. Any bipartite state can be written as $\rho_{AB} = \sum_{i,j} |i\rangle\langle j|_A \otimes O^B_{ij}$, suppose we are applying the measurement just on the subsystem A, therefore after the interaction with the measurement apparatus the state will be:

$$\tilde{\rho}_{AB:M} = \sum_{i,j} |i\rangle\langle j|_A \otimes O^B_{ij} \otimes |i\rangle\langle j|_M, \tag{4.41}$$

as the measure of entanglement is monotone by definition:

$$E(\tilde{\rho}_{AB:M}) \geq E(\tilde{\rho}_{B:M}) = E(\rho_{A:B}). \tag{4.42}$$

□

To compare two measures of different quantities, in our case we are comparing quantumness of correlation with quantum entanglement, it is necessary a common rule, the activation protocol gives the rule to compare these two quantities and this rule says that the measures of quantumness of correlations and quantum entanglement must be related via Eq.4.39.

4.3 Witnessed entanglement and geometrical measure of discord

In this section we will calculate a hierarchy relation between entanglement and quantumness of correlation following a geometrical approach. The measure of quantumness of correlation used will be the geometrical quantum discord via Schatten-p norm and we will quantify the entanglement via the witnessed entanglement description. Our motivation to study the relation between entanglement and geometrical discord follows from the following conjecture [66]:

Conjecture 111. *Given a density matrix $\rho_{AB} \in \mathcal{D}(\mathbb{C}^m \otimes \mathbb{C}^n)$, where $\min\{m, n\} = p$. Then the geometrical quantum discord via Hilbert-Schmidt norm is always greater or equal to the square of the negativity:*

$$D_{(2)}(\rho_{AB}) \geq \frac{4}{p(p-1)} \mathcal{N}^2(\rho_{AB}). \tag{4.43}$$

This bound for the Hilbert-Schmidt geometric discord has been proved for bipartite pure states, two-qubit states in general, and some bipartite Bell-diagonal states [66]. We can understand the validity of this conjecture for these cases and why it does not work in general via the following theorem [47,48].

Theorem 112. *Given a density matrix $\rho_{AB} \in \mathcal{D}(\mathbb{C}^m \otimes \mathbb{C}^n)$, the geometrical discord for any Schatten-p norm $D_p(\rho_{AB})$ and the witnessed entanglement $E_w(\rho_{AB})$ of ρ_{AB} satisfy the inequality:*

$$D_{(p)}(\rho_{AB}) \geq \left(\frac{E_w(\rho_{AB})}{\|W_\rho\|_q}\right)^p, \tag{4.44}$$

where $1/p + 1/q = 1$.

Proof. Given two operators A,B, which act on the same finite dimensional vector space $\mathbb{C}^d$ and the Schatten p-norm $\|A\|_p = Tr[(AA^\dagger)^{p/2}]^{1/p}$, then follow the Holder inequality:

$$\|A\|_p \|B\|_q \geq |Tr[AB^\dagger]|, \tag{4.45}$$

where $1/q + 1/p = 1$. The geometrical discord for a state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is:

$$D_p(\rho_{AB}) = \|\rho - \bar{\xi}\|_p^p, \tag{4.46}$$

where $\bar{\xi}$ is the closest non-discordant state. The witnessed entanglement of ρ can be written as:

$$E_w(\rho_{AB}) = \max\{0, -Tr[W_\rho \rho]\}, \tag{4.47}$$

where W_ρ is an optimal entanglement witness of ρ_{AB} for a given measure of entanglement. Plugging $A = \|\rho - \bar{\xi}\|_p^p$ and $B = W_\rho$ in Eq.4.47, we have:

$$\|\rho - \bar{\xi}\|_p \|W_\rho\|_q \geq |Tr[(\rho - \bar{\xi})W_\rho]|. \tag{4.48}$$

If ρ is entangled and $\bar{\xi}$ is separable we have $|Tr[(\rho - \bar{\xi})W_\rho]| \geq |Tr[\rho W_\rho]|$, thus:

$$\|\rho - \bar{\xi}\|_p \geq \frac{|Tr[\rho W_\rho]|}{\|W_\rho\|_q}, \tag{4.49}$$

which in terms of geometric discord in Eq.4.46 reads:

$$D_{(p)}(\rho) \geq \left(\frac{E_w(\rho)}{\|W_\rho\|_q}\right)^p. \tag{4.50}$$

□

As the witnessed entanglement is a measure of entanglement well defined also for multipartite systems and the geometrical discord can be extended to multipartite states just optimizing over the classical correlated states on the multipartite space, then the generalization of this bound for multipartite cases follow directly from Eq.4.44. This expression is interesting because we can compare the quantum discord - calculated geometrically - with any measure of entanglement which can be written in the witnessed entanglement form. We cannot guarantee that this bound is tight for every entanglement witness because the values of the entanglement for a given state ρ follow an hierarchy relation and each measure of entanglement has the optimal entanglement witness living in a different domain [23,25].

For the case where the geometrical discord is calculated via the Hilbert-Schmidt norm:

$$D_{(2)}(\rho_{AB}) = \min_{\xi \in \Omega_{Q.C.}} ||\rho_{AB} - \xi_{AX}||_2^2, \tag{4.51}$$

where $\xi_{AX} = \sum_x p_x \rho_x \otimes |x\rangle\langle x|$ and $\rho_{AB}, \xi_{AX} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, we have in Eq.4.44 $p = 2$ and $q = 2$, therefore:

$$D_{(2)}(\rho) \geq \left(\frac{E_w(\rho)}{\|W_\rho\|_2} \right)^2, \tag{4.52}$$

where $\|W_\rho\|_2^2 = Tr[W_\rho^2]$. We can take as an example the entanglement witness of the Negativity. The Negativity, $\mathcal{N}(\rho) \equiv (\|\rho^{T_A}\|_{(1)} - 1)/2$, can also be expressed in terms of witnessed entanglement as [25]:

$$\mathcal{N}(\rho) = \max\{0, -\min_{0 \leq W^{T_A} \leq \mathbb{I}} Tr(W\rho)\}. \tag{4.53}$$

The optimal entanglement witness for the negativity is given by the partial transpose of the projector composed by the eigenvectors of the partial transpose of ρ_{AB} related to the negative eigenvalues, therefore we can name $W_\rho = P_-^{T_A}$. The Hilbert-Schmidt norm of the optimal entanglement witness will be:

$$\|W_\rho\|_2^2 = Tr[W_\rho^2] = Tr[(P_-^{T_A})^2] = Tr[P_-^2] = Tr[P_-], \tag{4.54}$$

which is just the number of negative eigenvalues of the partial transpose of ρ_{AB}. For a pure state $|\psi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{r} c_i |a_i\rangle |b_i\rangle, \tag{4.55}$$

where r is the Schmidt number, the number of negative eigenvalues of the partial transpose of the state is given by the following proposition:

Proposition 113. *For a pure state with Schmidt decomposition as written in Eq.4.55, the spectrum of $\rho^{T_A} = \mathbb{I} \otimes T(|\psi\rangle\langle\psi|)$ is: $\{c_i c_i, i = 1, ..., r\} \cup \{\pm c_i c_j, i < j = 1, ..., r\}$ and for $p = \min(m, n)$ there are $p(|m-n| + p) - r^2$ eigenvalues equal to zero.*

Proof. Taking the partial transpose of $|\psi\rangle$:

$$\mathbb{I} \otimes T(|\psi\rangle\langle\psi|) = \sum_{i,j=1}^{r} c_i |a_i\rangle\langle a_j| \otimes |b_j\rangle\langle b_i|, \tag{4.56}$$

this matrix can be diagonalized in blocks, we have inside it some matrices which can be diagonalized easily, we have one diagonal matrix in the form:

$$\begin{pmatrix} c_1c_1 & \cdots & 0 & \cdots & 0 \\ & \ddots & & & \\ 0 & \cdots & c_ic_i & \cdots & 0 \\ & & & \ddots & \\ 0 & \cdots & 0 & \cdots & c_rc_r \end{pmatrix}, \tag{4.57}$$

whose eigenvalues are $\{c_i c_i, i = 1, ..., r\}$ and the respective eigenvectors are $\{|a_i\rangle |b_i\rangle\}$. We also have $r(r-1)/2$ 2×2 matrices in the form:

$$\begin{pmatrix} 0 & c_{i+1}c_i \\ c_{i+1}c_i & 0 \end{pmatrix}, \tag{4.58}$$

whose eigenvalues are $\{\pm c_{i+1} c_i\}$ and the eigenvectors are $\{|a_i + 1\rangle |b_i\rangle \pm |a_i\rangle |b_i + 1\rangle\}$. The remaining eigenvalues are zero, as the matrix is $mn \times mn$, we have $mn - r^2$ zero eigenvalues, and as $p = \min(m, n)$ then $mn = p(|m-n| + p)$, which proves the proposition. □

Therefore the partial transpose of a pure state has r negative eigenvalues, which is the Schmidt number. As $r \leq p = \min(m, n)$ the number of negative eigenvalues of the partial transpose are at most the dimension of the subsystem with the smallest dimension. Hence we can rewrite the Eq.4.52 for the Negativity of pure states as:

$$D_{(2)}(\rho) \geq \frac{\mathcal{N}^2(\Psi)}{p}, \tag{4.59}$$

where $\Psi = |\psi\rangle\langle\psi|$ and $p = \min(m, n)$. For systems with the same dimensions and $m = n = d > 3$, the bound in Eq.4.52 is tighter than the bound in the Conjecture.111.

For a general density matrix $\rho \in \mathcal{D}(\mathbb{C}^m \otimes \mathbb{C}^n)$, with an arbitrary rank, the spectrum of the partial transposed matrix $\mathbb{I} \otimes T(\rho)$ in function of the eigenvalues of ρ is not known, nor the number of negative eigenvalues in function of the rank or the dimension. However the number of negative eigenvalues is bounded by the dimension of the density matrix via the following theorem:

Theorem 114. *The number of negative eigenvalues of the partial transpose of $\rho \in \mathcal{D}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is always less than $(m-1)(n-1)$ [3, 135]*

Proof. A straightforward proof can be found on *theorem 1*, in the reference [135]. The idea to prove that the matrix $\mathbb{I} \otimes T(\rho)$ cannot have $(m-1)(n-1)+1$ negative eigenvalues. □

The proof that this bound is tight is not known, although numerically it was proved for some cases [88, 135]. Therefore we can rewrite the inequality Eq.4.52 as:

$$D_{(2)}(\rho) \geq \frac{\mathcal{N}^2(\rho)}{(m-1)(n-1)}. \tag{4.60}$$

Then we can compare this bound with the Conjecture.111 and we realize that we cannot guarantee that the conjecture is valid at all if the bipartite system has one dimension much greater than the other, for example if the system has dimension $2 \otimes n$ we have $2\mathcal{N}^2(\rho) \geq \frac{\mathcal{N}^2(\rho)}{(n-1)}$. Indeed a counter-example for the conjecture was showed by S. Rana and P. Parashar for a system with dimension $2 \otimes 3$ [136]. In that work they claim that the entanglement cannot be a lower bound for geometrical discord, however they just showed a counterexample for the conjecture. The entanglement can be a lower bound for geometrical discord as showed in Eq.4.44, the important point is: for geometrical discord calculated via Hilbert-Schmidt norm, the Hilbert-Schmidt norm of the witnessed entanglement must be take into account, then we have a rule and we can compare.

Another interesting measure of discord to discuss is the geometrical discord calculated via the trace norm. For a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ the discord will be:

$$D_{(1)}(\rho) = \min_{\xi \in \Omega_{Q.C.}} ||\rho_{AB} - \xi_{AX}||_1. \tag{4.61}$$

For this measure of discord we can calculate a linear bound between quantum discord and witnessed entanglement from Eq.4.44, as for $p = 1$ we have $q = \infty$:

$$D_{(1)}(\rho) \geq \frac{E_w(\rho)}{||W_\rho||_\infty}, \tag{4.62}$$

as we discussed in Chap.3 the trace-norm is the only norm such that the Schatten-p discord is an acceptable measure of the quantumness of correlations. This bound is interesting because the trace-norm is the best norm to distinguish the states, therefore we can write it linearly for the random robustness of entanglement. As discussed in Chap.3 the random robustness $\mathcal{R}_r(\rho)$ quantifies the resilience of the entanglement to white noise [158], and its witnessed entanglement is given by [23]:

$$\mathcal{R}_r(\rho) = \frac{1}{d} \max(0, -\min_{\{W \in \mathcal{W} \,|\, Tr(W)=1\}} Tr(W\rho)). \tag{4.63}$$

The random robustness is a subclass of the generalized robustness, with an extra restriction on the domain of the optimal entanglement witness $Tr(W_{RR} = 1)$. As the optimal entanglement witness of the generalized robustness lives in the domain $W_{GR} \leq \mathbb{I}$ [25], which implies that the biggest eigenvalue of the optimal entanglement witness of the random robustness must be between $[-1, 1]$, in other words, the domain of the optimal entanglement witness for the random robustness must satisfy $||W_{RR}||_\infty \leq 1$. In this way the infinity norm of the entanglement witness in the numerator of the Eq.4.62 contributes to the inequality to be tighter. Which implies that the inequality can be tight. An example is for $2 \otimes 2$ systems [48]. This property of the entanglement witness of the random robustness also implies that we can rewrite Eq.4.62 in a linear form:

$$D_{(1)}(\rho) \geq \frac{E_{\mathcal{R}_r}}{||W_{RR}||_\infty} \geq E_{\mathcal{R}_r}. \tag{4.64}$$

In last section we showed via the activation protocol that each measure of quantumness of correlation has an analogous measure of entanglement which is a linear lower bound for the quantumness of correlation. In Eq.4.64 we calculated a linear lower bound for the measure of quantumness of correlations related to the witnessed entanglement of the random robustness. This can be interesting because the witnessed entanglement of the random robustness can be calculated numerically via semidefinite programs [26, 27], which make comparison between quantumness of correlation and entanglement calculable. The bounds obtained in this section are a consequence of the geometry of the Hilbert space. To compare two different physical properties is always necessary a rule to the comparison makes sense, therefore we compare a geometrical measure of quantumness of correlation, calculated via Schatten-p norm, with a geometrical measure of entanglement, calculated via the entanglement witness formalism.

Chapter 5

Exploring the embedding in the context of local measurements

In this chapter we discuss the comparison between measurements via POVM and via projective measurements in the context of accessible information and discrimination of quantum states. We shall discuss that the former is related to the quantum discord of classical-quantum states, then the results obtained for it can be applied for the quantum discord. For the later we propose a dephased-POVM to obtain the projective measurement in this context. Based in this definition we introduce the concept of projectiveness of a given POVM, which is the shortest distance between the POVM and its dephased version. As an ensemble of quantum states $\xi = \{p_x, \rho_x\}_x$ is prepared according to a some classical random variable X, we can approach the measurement on an ensemble as a local measurement on a composed system. This system is composed by the classical random variable and the quantum system prepared according it:

$$\rho_{XS} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x, \tag{5.1}$$

where $\rho_X = \sum_x p_x |x\rangle\langle x|$ is the state of the classical random variable X, and $\rho_S = \sum_x p_x \rho_x$ is the state of the system. Therefore the ability to extract information and discriminate the states of an ensemble is intrinsically related to the quantumness of correlations of the state composed by the classical random variable and the quantum system prepared according it. Then we also discuss the comparison between POVM and projective measurement in the optimization of quantumness of correlations, first we discuss the quantum discord and in the end of the chapter the 1-way work deficit.

5.1 Accessible information: POVM vs Projective Measurement

In this section we shall introduce the concept of accessible information, and the amount of information which can be extracted from a finite ensemble of states, via a given POVM or a projective measurement. A particular class of ensemble which we will discuss is the G-covariant ensemble, which is generated via the action of the unitary representation of a group on the vector space. Finally we study an ensemble generated via the action of the $\mathcal{Z}_M$ group on 2-dimensional real space. We shall compare the information which can be extracted via projective measurement and via POVM.

5.1.1 Accessible information for real symmetric states

In this section we will illustrate a special kind of ensemble, the so called G-covariant ensemble. This class of ensembles is generated via the action of a group G on the Hilbert space. We are interested in these ensembles because the POVM which optimizes the accessible information has some friendly properties which permit us to calculate analytically a 2-dimensional ensemble of real states. We also compare the optimal strategy to maximize the accessible information with the optimal accessible information restricted to projective measurements. Let us first introduce the concept of G-covariant ensembles.

Definition 115. *A finite ensemble ξ of states is named G-covariant, or invariant under the action of a group G, if there is a unitary representation U_g of G such that for all states $\rho_g = U_g \rho U_g^{-1}$ is in the ensemble whenever ρ is in the ensemble.*

For a G-covariant ensemble, the POMV which optimizes the accessible information is also G-covariant [45, 138].

Theorem 116. *Let $\xi_M = \{p_i, \rho_i\}_{i=0,...,M-1}$ be an ensemble of real states, $\rho_i \in \mathbb{R}^d$, if it is G-covariant under the action of the irreducible unitary representation $U_k \in U(\mathbb{C}^d)$ of G with size $|G| = M \geq d$, there exists a state $|\alpha\rangle \in \mathbb{R}^d$, such that the POVM $\mathcal{A}$, with elements*

$$A_k = \frac{d}{M} U_k |\alpha\rangle\langle\alpha| U_k^\dagger, \tag{5.2}$$

where $k = 0, ..., M-1$, optimizes the accessible information.

Proof. Given $\mathcal{A} = \{A_l\}_{l=1,...,n}$ a rank-1 POVM which represents an optimal strategy to extract information from ξ_M, taking just the real part of $\mathcal{A}$, $\tilde{\mathcal{A}} = \mathrm{Re}(\mathcal{A}) = \{\tilde{A}_l\}_{l=1,...,n}$, we have:

- as the elements A_i are positive, $\tilde{A}_i = \frac{A_i + A_i^\dagger}{2}$ is positive too;
- as $\sum_i A_i = \mathbb{I}$, then $\sum_i \tilde{A}_i = \sum_i \frac{A_i + A_i^\dagger}{2} = \mathbb{I}$;
- given a state $\rho \in D(\mathbb{C}^d)$, the probability elements remain the same:

$$Tr[\tilde{A}_i \rho] = Tr\left[\frac{A_i + A_i^\dagger}{2}\rho\right] = \frac{Tr[A_i\rho] + Tr[A_i^\dagger \rho]}{2} = Tr[A_i\rho]. \tag{5.3}$$

Therefore the POVM $\tilde{\mathcal{A}}$ remains an optimal measurement for accessible information, although we cannot guarantee it remains rank-1. Decomposing $\tilde{\mathcal{A}}$ in a rank-1 POVM $\mathcal{B} = \{B_k\}_{k=1,...,m}$, for $m \geq n$, in this way each element $A_i = \sum b_k B_{ki}$. We can express the accessible information via the Shannon relative entropy (2):

$$I(\xi_M) = I(\xi_M : \mathcal{A}) = H(p(x,y)||p_x p(y)), \tag{5.4}$$

given

$$\begin{aligned} p(x,y) &= p_x Tr[A_y \rho_x] = p_x \sum_z b_z Tr[B_{zy}\rho_x]; && (5.5) \\ p(y) &= \sum_x \sum_z p_x b_z Tr[B_{zy}\rho_x], && (5.6) \end{aligned}$$

then

$$\begin{aligned} H(p(x,y)||p_x p(y)) &= H\left(\sum_z b_z p_x p(zy|x) || \sum_z b_z p_x p(zy)\right) && (5.7) \\ &\leq \sum_z b_z H(p_x p(zy|x)||p_x p(zy)) && (5.8) \\ &= \sum_z b_z \sum_x p_x \sum_y p(zy|x) \log\left(\frac{p_x p(zy|x)}{p_x p(zy)}\right) && (5.9) \\ &= \sum_x p_x \sum_z \sum_y b_z p(zy|x) \log\left(\frac{b_z p_x p(zy|x)}{b_z p_x p(zy)}\right) && (5.10) \\ &= I(\xi_M : \mathcal{B}). && (5.11) \end{aligned}$$

In Eq.5.8 we used the convexity of the relative entropy, in the Eq.5.9 we wrote the relative entropy by the definition (2). As the accessible information is given by the optimal measurement, the POVM $\mathcal{B}$ must be an optimal measurement too.

The POVM $\mathcal{B}$ is composed by rank-1 real elements. We can define another POVM $\mathcal{C}$ with $M \cdot m$ elements:

$$C_{kg} = \frac{1}{M} U_g B_k U_g^{\dagger}. \tag{5.12}$$

Now we must check that $I(\xi_M : \mathcal{B}) = I(\xi_M : \mathcal{C})$. The conditional probability to get an outcome C_{kg} will be:

$$p(kg|i) = Tr[C_{kg}\rho_i] = \frac{1}{M} Tr[U_g B_k U_g^{\dagger} \rho_i], \tag{5.13}$$

as the set of states ρ_i is invariant under the action of the group G:

$$\sum_i p_i p(kg|i) = \frac{1}{M} \sum_i p_i Tr[U_g B_k U_g^{\dagger} \rho_i] = \frac{1}{M} \sum_i p_i Tr[B_k (U_g \rho_i U_g^{\dagger})^{\dagger}], \tag{5.14}$$

thus the action of the group G will imply in just a reordering in the sum, thus

$$\sum_i p_i p(kg|i) = \frac{1}{M} \sum_i p_i Tr[B_k \rho_i] = \frac{1}{M} \sum_i p_i p(k|i). \tag{5.15}$$

Therefore taking the mutual information:

$$\begin{aligned} I(\xi_M : \mathcal{C}) &= \sum_i p_i \sum_{kg} p(kg|i) \log \left(\frac{p(kg|i)}{\sum_l p_l p(kg|l)} \right) && (5.16) \\ &= \sum_i p_i \sum_{kg} Tr[C_{kg}\rho_i] \log \left(\frac{Tr[C_{kg}\rho_i]}{\sum_l p_l p(k|l)/M} \right) && (5.17) \\ &= \sum_i p_i \sum_{kg} Tr[B_k (U_g \rho_i U_g^{\dagger})^{\dagger}] \log \left(\frac{Tr[B_k (U_g \rho_i U_g^{\dagger})^{\dagger}]}{\sum_l p_l p(k|l)/M} \right) && (5.18) \\ &= I(\xi_M : \mathcal{B}). && (5.19) \end{aligned}$$

Thus $\mathcal{C}$ remains optimal. Finally, for each $B_l / Tr[B_l]$ we have another POVM $\mathcal{D}_l$, as enunciated by the Schur's lemma, with elements $D_g^l = \frac{d}{M} U_g B_l U_g^{\dagger} / Tr[B_l]$. Hence the POVM $\mathcal{C}$ can be written as a convex combination of $\mathcal{D}_l$:

$$\mathcal{C} = \bigoplus_i \frac{Tr[B_i]}{d} \mathcal{D}_i, \tag{5.20}$$

therefore via Proposition 48:

$$I(\xi_M : \mathcal{C}) \leq \sum_i \frac{Tr[B_i]}{d} I(\xi_M : \mathcal{D}_i) \leq \max_i I(\xi_M : \mathcal{D}_i). \tag{5.21}$$

As $\mathcal{C}$ is optimal, at least one $\mathcal{D}_i$ is optimal, which proves the theorem. □

This theorem was enunciated and proved restricted to the real space, although it is also valid when the representation of the group acts irreducibly in the whole complex space [45]. We just proved for the real space because we shall study an ensemble composed by real symmetric states, in this way ensuring the validity of the description hereafter. If the representation of the group is not irreducible this theorem is not valid. In this case we must take in account the number of reductions, and we have one POVM for each irreducible representation. The optimal POVM will be a convex combination of these POVMs [50]. P. Shor calculated numerically the accessible information for a G-covariant ensemble of real states in a 3-dimensional space, and he has shown that in order to attain the accessible information is necessary a convex combination of two POVMs with 3 elements each [144].

Lemma 117. *The information extracted from a covariant ensemble of equiprobable states* $\xi_M = \{1/M, \rho_x\}_{x=0,\dots,M-1}$ *and a covariant POVM* $\mathcal{A}$ *with elements* $A_y = \frac{d}{M} U_y |\alpha\rangle\langle\alpha| U_y^\dagger$*, for a given* $|\alpha\rangle \in \mathbb{C}^d$ *can be written as:*

$$I(\xi_M : \mathcal{A}) = \log d + \frac{d}{M} \sum_{x=0}^{M-1} \langle\alpha| \rho_x |\alpha\rangle \log \langle\alpha| \rho_x |\alpha\rangle. \tag{5.22}$$

Proof. Given the conditional probability $p(x|y) = Tr[\rho_x A_y]$ the joint probability will be:

$$p(x,y) = \frac{1}{M} Tr[\rho_x A_y] \tag{5.23}$$

$$= \frac{d}{M^2} Tr[\rho_x U_y |\alpha\rangle\langle\alpha| U_y^\dagger] \tag{5.24}$$

$$= \frac{d}{M^2} Tr[(U_y \rho_x U_y^\dagger)^\dagger |\alpha\rangle\langle\alpha|]. \tag{5.25}$$

Therefore the probability to get the output x given $\xi = \frac{1}{M}\sum_x U_x \rho U_x^\dagger$ is :

$$p(y) = \sum_x p(x,y) = \frac{d}{M^2} \sum_x Tr[(U_y \rho_x U_y^\dagger)^\dagger |\alpha\rangle\langle\alpha|], \tag{5.26}$$

as the action of the group in the states will just reorder the sum, hence:

$$\sum_x p(x,y) = \frac{d}{M^2} \sum_x Tr[U_x^\dagger \rho U_x |\alpha\rangle\langle\alpha|], \tag{5.27}$$

as A_y are POVM elements, $\sum_y A_y = \sum_y \frac{d}{M} U_y |\alpha\rangle\langle\alpha| U_y^\dagger = \mathbb{I}$, hence:

$$\sum_x p(x,y) = \frac{1}{M} Tr[\rho] = \frac{1}{M}. \tag{5.28}$$

In this way we have $\sum_x p(x,y) = \sum_y p(x,y) = 1/M$.

Taking the Shannon mutual information for the joint probability distribution $\{p(x,y)\}_{x,y=0,\dots,M-1}$:

$$H(X:Y) = H(X) + H(Y) - H(X,Y) \tag{5.29}$$

$$H(X:Y) = 2\log(M) + \sum_{x,y} \frac{d}{M^2} \langle\alpha| U_y^\dagger \rho_x U_y |\alpha\rangle \log\left[\frac{d}{M^2} \langle\alpha| U_y^\dagger \rho_x U_y |\alpha\rangle\right], \tag{5.30}$$

for each element y the action of the group will reorder the sum over x, therefore we can relabel the sum such that:

$$H(X:Y) = 2\log(M) + M\sum_x \frac{d}{M^2} \langle\alpha| \rho_x |\alpha\rangle \log\left[\frac{d}{M^2} \langle\alpha| \rho_x |\alpha\rangle\right], \tag{5.31}$$

$$H(X:Y) = \log(d) + \frac{d}{M} \sum_x \langle\alpha| \rho_x |\alpha\rangle \log[\langle\alpha| \rho_x |\alpha\rangle]. \tag{5.32}$$

□

Given a density matrix $\zeta \in \mathcal{D}(\mathbb{C}^d)$, we have a set of states $U_g \zeta U_g^\dagger$ covariant under the action of the group G, as the unitary representation of this group $\{U_g\}_{g=0,\dots,M-1}$ is irreducible in the space $\mathbb{C}^d$, by the Schur's lemmas an operator which commutes with all elements of the irreducible representation will be a constant times the identity in $\mathbb{C}^d$. It is straightforward to check that the operator $K = \sum_g U_g \zeta U_g^\dagger$ commutes with the matrices $\{U_h\}_{h=0,\dots,|G|-1}$:

$$\begin{aligned} KU_h &= \sum_g U_g \zeta U_g^\dagger U_h = \sum_g \overbrace{U_h U_h^\dagger}^{\mathbb{I}} U_g \zeta U_g^\dagger U_h & (5.33)\\ &= \sum_g U_h U_{gh^{-1}} \zeta U_{gh^{-1}}^\dagger = U_h \sum_l U_l \zeta U_l^\dagger & (5.34)\\ &= U_h K. & (5.35)\end{aligned}$$

Therefore we have $K = \sum_g U_g \zeta U_g^\dagger = \lambda \mathbb{I}$, and we can determine λ taking the trace of K:

$$\begin{aligned} Tr[K] = \sum_g Tr[U_g \zeta U_g^\dagger] &= |G| Tr[\zeta] & (5.36)\\ &= Tr[\lambda \mathbb{I}] = d\lambda, & (5.37)\end{aligned}$$

hence $\lambda = \frac{|G|}{d}$ and:

$$\sum_g U_g \zeta U_g^\dagger = \frac{|G|}{d} \mathbb{I}, \tag{5.38}$$

which also guarantees that the set $A_y = \frac{d}{M} U_y |\alpha\rangle\langle\alpha| U_y^\dagger$ is a set of elements of a POVM. This also gives the following property for the states $\xi = \frac{1}{|G|}\sum_g U_g \rho U_g^\dagger \in \mathcal{D}(\mathbb{C}^d)$:

$$\xi = \frac{1}{|G|} \sum_g U_g \rho U_g^\dagger = \frac{\mathbb{I}}{d}. \tag{5.39}$$

Therefore the action of the irreducible representation of a group gives a set of states symmetric enough to the convex combination of them, with the same probability, be the maximal mixture state in the representation space.

Case of Study: $d = 2$

An example of group which we can study is the real symmetric permutation group $\mathbb{Z}_M$, which is the set of integer numbers with the operation of sum modulo M. The accessible information of the covariant ensemble generated via this group in a 2-dimensional Hilbert space was investigate by A. Peres and W. Wotters [125], where the authors studied the incapacity to access the information performing local measurements. Curiously this work was the seed for the development of the *quantum teleport* [17]. Another interesting study about this ensemble was performed by C. Bennett *et. al* [18], where the authors discuss quantum correlations and nonlocal action without the presence of entanglement between the parts.

Considering the ensemble of real equiprobable states $\xi = \{1/M, |\psi_k\rangle\langle\psi_k|\}$, where $|\psi_k\rangle \in \mathbb{R}^d$ are generated via the action of the unitary representation $\{U_k\}_{k=0,\dots,M-1}$ of the group $\mathbb{Z}_M$. For this class of states, the irreducibility of this representation is restricted to the real space, thus $U_k \in \mathcal{U}(\mathbb{R}^d)$. For a 2-dimensional Hilbert spaces, this representation is given by $U_k = \exp(-i\pi\sigma_y k/M)$, where σ_y is the Pauli matrix:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.40}$$

As this representation group is irreducible over the action on the real space, the covariant ensemble must be a set of real states, which in the Bloch sphere is the XZ-plane:

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_k\rangle = U_k |\psi_0\rangle = \begin{pmatrix} \cos(\pi k/M) \\ \sin(\pi k/M) \end{pmatrix}. \tag{5.41}$$

For $M = 2$, it is clear that the states $|\psi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\psi_0\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ are orthogonal, then for any M =even this orthogonality symmetry holds. It is enunciated in the following proposition:

Proposition 118. *If the number of states in the ensemble $\xi = \{1/M, |\psi_k\rangle\langle\psi_k|\}$ is even, where the states $|\psi_k\rangle$ are given in 5.41, the optimal strategy to access the information is a projective measurement.*

Proof. This occurs because the ensemble is composed by $M/2$ pairs of orthogonal states. Given the state written as a convex combination of the states in the ensemble with equal probability:

$$\rho = \frac{1}{M}\sum_{k=0}^{M-1}|\psi_k\rangle\langle\psi_k| = \frac{1}{M}\sum_{k=0}^{M/2-1}|\psi_k\rangle\langle\psi_k| + \frac{1}{M}\sum_{k=M/2}^{M-1}|\psi_k\rangle\langle\psi_k| = \rho_0 + \rho_{M/2}, \tag{5.42}$$

where

$$\rho_{M/2} = \frac{1}{M}\sum_{k=M/2}^{M-1}|\psi_k\rangle\langle\psi_k| = \frac{1}{M}\sum_{k=0}^{M/2-1}|\psi_{k+M/2}\rangle\langle\psi_{k+M/2}|, \tag{5.43}$$

therefore the states in the ensemble will be

$$|\psi_{k+M/2}\rangle = \begin{pmatrix}\cos[\pi(k+M/2)/M]\\ \sin[\pi(k+M/2)/M]\end{pmatrix} = \begin{pmatrix}\cos[\pi k/M+\pi/2]\\ \sin[\pi k/M+\pi/2]\end{pmatrix} = \begin{pmatrix}-\sin(\pi k/M)\\ \cos(\pi k/M)\end{pmatrix}, \tag{5.44}$$

where $\langle\psi_0|\psi_{M/2}\rangle = 0$, hence the states ρ_0 and $\rho_{M/2}$ have orthogonal projections, in other words for each pure state in the state ρ_0 there is another pure state in $\rho_{M/2}$ orthogonal to it. By the symmetry of the states, each orthogonal pair forms an optimal projective measurement. □

This symmetry will appear again when we compare the accessible information with the information extracted from an ensemble restricted to projective measurement, we will see that for M even this information is the accessible information.

For a covariant ensemble, given the theorem 116, the elements of the POVM $\mathcal{A}$ which optimizes the accessible information are in the form:

$$A_k = \frac{2}{M}U_k|\alpha\rangle\langle\alpha|U_k^\dagger, \tag{5.45}$$

such that given $U_k = \exp(-i\pi k\sigma_y/M)$ and $|\alpha\rangle$ in the XZ-plane, elements of the POVM $\mathcal{A}$ are the vectors

$$|\alpha_k\rangle = \sqrt{\frac{2}{M}}U_k|\alpha\rangle = \sqrt{\frac{2}{M}}\begin{pmatrix}\cos(\theta+\pi k/M)\\ \sin(\theta+\pi k/M)\end{pmatrix},$$

where θ is the angle between the vector $|\alpha\rangle$ and the Z-axis in the Bloch sphere. In this way we can write the information extracted from the ensemble ξ_M in function of θ and find the angle which optimizes the accessible information. The conditional probability $p(y|x) = |\langle\alpha_y|\psi_x\rangle|^2$ to give the output $|\alpha_y\rangle\langle\alpha_y|$ given the input state $|\psi_x\rangle\langle\psi_x|$ will be:

$$\begin{aligned} p(y|x) &= |\langle\alpha_y|\psi_x\rangle|^2 && (5.46)\\ &= \left(\sqrt{\frac{2}{M}}[\cos(\theta+\pi y/M)\cos(\pi x/M)+\sin(\theta+\pi y/M)\sin(\pi x/M)]\right)^2 && (5.47)\\ &= \frac{2}{M}\cos[\theta-\pi(x-y)/M]^2 && (5.48)\\ &= \frac{2}{M}\left(\frac{1+\cos[2\theta-2\pi(x-y)/M]}{2}\right) && (5.49)\\ &= \frac{1}{M}(1+\cos[2\theta-2\pi(x-y)/M]), && (5.50)\end{aligned}$$

where $|\psi_x\rangle = \cos(x\pi/M)|0\rangle + \sin(k\pi/M)|1\rangle$ and $|\alpha_y\rangle = \sqrt{2/M}[\cos(\theta+\pi y/M)|0\rangle + \sin(\theta+\pi y/M)|1\rangle]$.

Therefore the joint probability is $p(x,y) = p_x p(y|x) = \frac{1}{M} p(y|x) = \frac{1}{M^2}\left[1+\cos\left[2\theta - 2\pi(x-y)/M\right]\right]$ and the marginal probability distribution $p(y) = \sum_x p(x,y)$ will be:

$$
\begin{aligned}
p(y) &= \frac{1}{M}\sum_x p(x,y) = \frac{1}{M^2}\sum_x \left[1+\cos\left(2\theta - 2\pi(x-y)/M\right)\right] && (5.51)\\
&= \frac{1}{M^2}\sum_l \left[1+\cos\left(2\theta - 2\pi l/M\right)\right] && (5.52)\\
&= \frac{1}{M^2}\sum_l \left[1+\cos\left(2\theta\right)\cos\left(2\pi l/M\right)+\sin\left(2\theta\right)\sin\left(2\pi l/M\right)\right] && (5.53)\\
&= 1/M, && (5.54)
\end{aligned}
$$

where in Eq.5.52 we relabeled the sum with $l = x - y$ and reordered the terms, in Eq.5.54 we used the trigonometric property of $\sum_k \cos \pi k/M = \sum_k \sin \pi k/M = 0$. Hence the information extracted from ξ_M with the POVM $\mathcal{A}$ can be written in function of θ as:

$$
\begin{aligned}
I(\theta) &= 2\log M + \frac{1}{M^2}\sum_{x,y}\left\{1+\cos\left[2\theta-2\pi(x-y)/M\right]\right\}\log\frac{1}{M^2}\left\{1+\cos\left[2\theta-2\pi(x-y)/M\right]\right\},\\
&= \frac{1}{M^2}\sum_{x,y}\left\{1+\cos\left[2\theta-2\pi(x-y)/M\right]\right\}\log\left\{1+\cos\left[2\theta-2\pi(x-y)/M\right]\right\}, && (5.55)\\
&= \frac{1}{M}\sum_{l}\left\{1+\cos\left[2\theta-2\pi l/M\right]\right\}\log\left\{1+\cos\left[2\theta-2\pi l/M\right]\right\}, && (5.56)
\end{aligned}
$$

where in Eq.5.55 we used the multiplication property of the log function and in Eq.5.56 we relabeled the sum with $l = x - y$ and used the same trick performed in Eq.5.54. The value of θ which optimizes the accessible information is ruled by the following proposition:

Proposition 119 (Optimal strategy for accessible information [138])**.** *For the ensemble of real symmetric states $\xi_M = \{1/M, |\psi_k\rangle\langle\psi_k|\}_{k=0,\ldots,M-1}$, where the states $|\psi\rangle \in \mathbb{R}^2$ are given by Eq.5.41, each element of the POVM which optimizes the accessible information is orthogonal with one state in the ensemble and it has the form $A_l = \frac{d}{M} U_l |\alpha\rangle\langle\alpha| U_l^\dagger$.*

Proof. Given Eq.5.56:

$$
I(\theta) = \frac{1}{M}\sum_l \left[1+\cos\left(2\theta-2\pi l/M\right)\right]\log\left[1+\cos\left(2\theta-2\pi l/M\right)\right], \tag{5.57}
$$

as $|\cos\left[2\theta - 2\pi l/M\right]| \leq 1$ we can expand $I(\theta)$ with the aid of the expression [1]

$$
(1+x)\log(1+x) = x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}x^n \quad |x| \leq 1, \tag{5.58}
$$

thus:

$$
\begin{aligned}
I(\theta) &= \frac{1}{M}\sum_l \left[\cos\left(2\theta-2\pi l/M\right) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}\cos\left(2\theta-2\pi l/M\right)^n\right], && (5.59)\\
&= \frac{1}{M}\sum_l \left[\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}\cos\left(2\theta-2\pi l/M\right)^n\right], && (5.60)\\
&= \frac{1}{M}\sum_l \left[\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2n(2n-1)}\cos\left(2\theta-2\pi l/M\right)^{2n} + \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{2n(2n+1)}\cos\left(2\theta-2\pi l/M\right)^{2n+1}\right],
\end{aligned}
$$

[1]This expression is the expansion for the natural logarithm, as we can change this expansion for the logarithm in the basis 2 just multiplying for $1/\ln 2$ in the series, although to clear the notation we shall omit this constant.

where in Eq.5.60 we used that $\sum_l \cos[2\theta - 2\pi l/M] = 0$ and in the last expression we separated the series in even and odd terms. Getting the power reduction formula for cosine:

$$\cos^m \omega = \frac{1}{2^m}\binom{m}{m/2} + \frac{2}{2^m}\sum_{k=0}^{m/2-1}\binom{m}{k}\cos(m-2k)\omega \quad \text{for } m \text{ even}, \tag{5.61}$$

$$\cos^m \omega = \frac{2}{2^m}\sum_{k=0}^{(m-1)/2}\binom{m}{k}\cos(m-2k)\omega \quad \text{for } m \text{ odd}, \tag{5.62}$$

hence for $\omega = 2\theta - 2\pi k/M$,

$$I(\theta) = \frac{1}{M}\sum_{l=0}^{M-1}\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{2n(2n-1)}\left\{\frac{1}{2^{2n}}\binom{2n}{n} + \frac{2}{2^{2n}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos[(2n-2k)(2\theta-2\pi k/M)]\right\} + $$
$$+ \frac{(-1)^{2n+1}}{2n(2n+1)}\left\{\frac{2}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\cos[(2n+1-2k)(2\theta-2\pi k/M)]\right\} \tag{5.63}$$

as we have

$$\sum_k \cos(2\theta - 2\frac{k\pi}{M}L) = \cos 2\theta L\sum_k \cos\frac{2\pi k}{M}L + \sin 2\theta L\sum_k \sin\frac{2\pi k}{M}L \tag{5.64}$$

and using the Lagrange's trigonometric identity:

$$\sum_{k=0}^{M-1}\cos\omega k = -\frac{1}{2} + \frac{\sin(M+1/2)\omega}{2\sin\omega/2}, \tag{5.65}$$

$$\sum_{k=0}^{M-1}\sin\omega k = \frac{1}{2}\cot(\omega/2) - \frac{\cos(M+1/2)\omega}{2\sin\omega/2}. \tag{5.66}$$

In our case we have $\omega = 2\pi L/M$, thus

$$\sum_{k=0}^{M-1}\cos 2\pi L/Mk = M\delta_{L,Mq}, \tag{5.67}$$

$$\sum_{k=0}^{M-1}\sin 2\pi L/Mk = 0, \tag{5.68}$$

for $q =$ integer. Replacing $L = 2n + 1 - 2k$ in the mutual information and applying the last two results we have:

$$I(\theta) = \sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{2n}{n} + \sum_{k=0}^{n}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\left[\frac{(-1)^{2n}}{2n(2n-1)}\frac{2}{2^{2n}}\binom{2n}{k}\delta_{2n+2k,qM} + \right. \tag{5.69}$$
$$\left. + \frac{(-1)^{2n+1}}{2n(2n+1)}\frac{2}{2^{2n+1}}\binom{2n+1}{k}\delta_{2n+2k-1,Mq}\right]\cos(2\theta Mq),$$

$$I(\theta) = \sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{2n}{n} + \sum_{q=0}^{\infty}f(qM)(-1)^{qM}\cos(2\theta Mq), \tag{5.70}$$

where

$$f(qM) = \sum_n\sum_k \frac{1}{2(2l+qM)(2l+qM-1)}\frac{1}{2(2l+qM-1)}\binom{2l+Mq}{k}(\delta_{2n+2k,qM} + \delta_{2n+2k-1,Mq}), \tag{5.71}$$

as $f(qM)$ does not depend on θ, the function $I(\theta)$ is maximized when

$$(-1)^{qM}\cos(2\theta Mq) = 1, \tag{5.72}$$

which as Mq is an integer is solved for $\theta = \pi/2$ for all M, thus the state $|\alpha\rangle = |1\rangle$. Therefore the POVM which optimizes the accessible information will be such that the elements are rank-1 and each one is orthogonal to the states in the ensemble:

$$\langle\alpha_l|\psi_l\rangle = 0 \quad \text{and} \quad \langle\alpha_l|\psi_k\rangle \neq 0 \quad \text{for} \quad k \neq l. \tag{5.73}$$

□

Physically this result means that when we get the output $|\alpha_j\rangle$ we can with certainly that the state $|\psi_j\rangle$ was not the input. The optimal strategy for accessible information is not unique, thus we can choose for example the optimal POVM with the smaller amount of elements. E. Davies showed that for an ensemble of real states the optimal POVM can be composed at least by $d(d-1)/2$ [45]. For the ensemble of real states 5.41 Sasaki et. al gave a recipe to construct an optimal POVM with 3 elements, which atain the Davies bound [138]. For an ensemble with 3 elements composed by 3-dimensional real symmetric states P. Shor calculated numerically an optimal POVM with 6 elements, although he used a reducible representation of the permutation group, thus the optimal POVM must be composed via convex combination of two POVMs, where which one has 3 elements, as we discussed [144]. For ensemble of states with more elements than the Davies bound the optimal strategy can be chosen based on the coding-decoding process to attain the channel capacity of the measurement map [45]. With this idea is possible to propose a measure of information power of measurement maps [41].

Substituting the optimal strategy on Eq.5.22, the accessible information will be:

$$I(\xi_M) = \log 2 + \frac{2}{M}\sum_{x=0}^{M-1} \sin^2 x\pi/M \log \sin^2 x\pi/M, \tag{5.74}$$

where $\langle\alpha|\rho_x|\alpha\rangle = (\langle\alpha|\psi_k\rangle)^2 = \sin^2 x\pi/M$. We shall calculate the maximal amount of information which can be extracted from ξ_M restricting the strategy to projective measurements. As the states in the ensemble are restricted to the XZ-plane in the Bloch sphere, we expect that the optimal projective measurement lives in the XZ-plane too, because components outside this plane do not contribute to the expectation values of the measurement observables. Hence we can restrict the projective measurement $\mathcal{P} = \{|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|\}$ to live in this plane:

$$|\phi_0\rangle = V|0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle \tag{5.75}$$

$$|\phi_1\rangle = V|1\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle, \tag{5.76}$$

where:

$$V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The conditional probability $p(y|x)$ to give the output ϕ_y when the input is the state $|\psi_x\rangle$ is defined as:

$$p(y|x) = |\langle y|V|\psi_x\rangle|^2, \tag{5.77}$$

thus we have the joint probability distribution $p(x,y) = 1/M|\langle y|V|\psi_x\rangle|^2$, the marginal distribution will be:

$$p_x = \sum_{y=0}^{1} P(x,y) = 1/M \langle\psi_x| \left(\sum_y V|y\rangle\langle y| V^\dagger\right) |\psi_x\rangle = 1/M \tag{5.78}$$

$$p(y) = \sum_{x=0}^{M-1} P(x,y) = 1/M \langle y| V^\dagger \left(\sum_x |\psi_x\rangle\langle\psi_x|\right) V|y\rangle = 1/2. \tag{5.79}$$

The Shannon entropy for the joint probability and the marginal distributions will be:

$$H(X) = \log M \tag{5.80}$$

$$H(Y) = \log 2 \tag{5.81}$$

$$H(X,Y) = \log M - \frac{1}{M}\sum_{x,y} |\langle y| V|\psi_x\rangle|^2 \log(|\langle y| V|\psi_x\rangle|^2), \tag{5.82}$$

therefore the information extracted from ξ_M via the projective measurement $\mathcal{P}$ will be:

$$I(\xi_M : \mathcal{P}) = H(X) + H(Y) - H(X,Y) = \log 2 + \frac{1}{M}\sum_{x,y} |\langle y| V|\psi_x\rangle|^2 \log(|\langle y| V|\psi_x\rangle|^2), \tag{5.83}$$

where elements of the series in y are:

$$\langle 0| V^\dagger |\psi_k\rangle = \cos\theta\cos\pi k/M + \sin\theta\sin\pi k/M = \cos(\theta - \pi k/M) \tag{5.84}$$

$$\langle 1| V^\dagger |\psi_k\rangle = \cos\theta\sin\pi k/M - \sin\theta\cos\pi k/M = -\sin(\theta - \pi k/M) \tag{5.85}$$

as they are real we have:

$$\begin{aligned} I(\xi_M : \mathcal{P}) = \log 2 \quad &+ \quad \frac{1}{M}\sum_x (\cos(\theta - \pi k/M))^2 \log[(\cos(\theta - \pi k/M))^2] + \\ &+ \quad \frac{1}{M}\sum_x (\sin(\theta - \pi k/M))^2 \log[(\sin(\theta - \pi k/M))^2]. \end{aligned} \tag{5.86}$$

Now our challenge is to calculate the value of θ which maximizes Eq.5.86, thus follows the proposition:

Proposition 120 (Optimal projective measurement to extract information [49])**.** *The elements of the optimal projective measurement $P \in P(\mathbb{C}_B, \mathbb{C}_B)$ to extract the information from the ensemble of real symmetric states $\xi_M = \{1/M, |\psi_k\rangle\langle\psi_k|\}_{k=0,\dots,M-1}$, where $|\psi_k\rangle \in \mathbb{R}^2$ is given by*

$$|\psi_k\rangle = \cos(\pi k/M)|0\rangle + \sin(\pi k/M)|1\rangle.$$

Proof. We shall calculate the point of maximum using the same tricks used to get the optimum POVM in the Proposition.119. Substituting the following trigonometric identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \tag{5.87}$$

then replacing, just to clear the notation, $x_k = \theta - \pi k/M$ in Eq.5.86:

$$I(\xi_M : \mathcal{P}) = \frac{1}{2M}\sum_k (1 + \cos 2x_k)^2 \log[(1 + \cos 2x_k)^2] + \frac{1}{2M}\sum_k (1 - \cos 2x_k))^2 \log[(1 - \cos 2x_k))^2]. \tag{5.88}$$

Given again the series expansion:

$$(1 \pm x) \log (1 \pm x) = \pm x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} (\pm x)^n \quad |x| \leq 1, \tag{5.89}$$

substituting in Eq.5.88:

$$I(\xi_M : \mathcal{P}) = \frac{1}{2M} \sum_{k=0}^{M-1} \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n(n-1)} (\cos 2x_k)^n + \frac{(-1)^n}{n(n-1)} (-1)^n (\cos 2x_k)^n \right), \tag{5.90}$$

as the n =odd terms will cancel:

$$I(\xi_M : \mathcal{P}) = \frac{1}{2M} \sum_{k=0}^{M-1} \sum_{n=1}^{\infty} \left(\frac{(-1)^{2n}}{2n(2n-1)} (\cos 2x_k)^{2n} \right). \tag{5.91}$$

Given the power reduction formula Eq.5.61 for even power and summing over k:

$$\sum_k \cos^{(2n)} [2x_k] = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{2}{2^{2n}} \sum_{l=0}^{n-1} \binom{n}{l} \sum_k \cos [(2n-2l)2x_k], \tag{5.92}$$

repeating the same steps of calculus performed from Eq.5.65 to Eq.5.67, the mutual information will be:

$$I(\xi_M : \mathcal{P}) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} + \sum_{l=0}^{n} \sum_{n=1}^{\infty} \frac{2}{2^{2n}} \binom{2n}{l} \sum_{q=0}^{\infty} \cos (2\theta[2n-2l]) \delta_{2n-2l,Mq}. \tag{5.93}$$

The term which depends on θ is

$$\cos (2\theta[2n-2l]) \delta_{2n-2l,Mq} = \cos (2\theta Mq) \delta_{2n-2l,Mq},$$

as $Mq = 2n - 2l$ =even and M is odd implies q is odd, therefore

$$\cos (2\theta Mq) = \cos (2\theta 2p),$$

we attain the maximum when $\cos (2\theta 2p) = 1$, or $2\theta 2p = 2p\pi$, then the mutual information $I(\xi : \mathcal{P})$ is maximum for $\theta = 0$ or $\theta = \pi/2$, which implies that the projective measurement $\mathcal{P} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is the optimal strategy restricted to projective measurements. □

Substituting the optimal projective measurement strategy in Eq.5.86:

$$\begin{aligned} I(\xi_M)_\Pi = \log 2 \quad &+ \quad \frac{1}{M} \sum_k (\cos (\pi k/M))^2 \log [(\cos (\pi k/M))^2] + \\ &+ \quad \frac{1}{M} \sum_k (\sin (\pi k/M))^2 \log [(\sin (\pi k/M))^2]. \end{aligned} \tag{5.94}$$

Therefore we can compute the difference between the information extracted from the ensemble ξ_M via the optimal POVM and via the optimal projective measurement:

$$I(\xi_M) - I(\xi_M)_\Pi = \frac{1}{M} \sum_k \{ (\sin (\pi k/M))^2 \log [(\sin (\pi k/M))^2] - (\cos (\pi k/M))^2 \log [(\cos (\pi k/M))^2] \}, \tag{5.95}$$

where $I(\xi_M)_\Pi$ is the accessible information restricted to projective measurements given in Eq. 5.94 and $I(\xi_M)$ is the accessible information given in Eq.5.74. We can realize that it indeed is greater than zero, which means that even in qubit systems POVM can extract more information than projective measurements. As for M = even the Eq.5.94 is zero and it decreases when M grows, then the biggest value of the difference in Eq.5.94 is for $M = 3$. A general relation like obtained for 2-dimensional systems in Eq.5.94 is not easy to obtain, once that the G-covariant ensemble depends on the representation of the group G, and this representation depends on the dimension. The other important trouble, that we had in our "saga" to understand the difference between the restriction to projective measurements in the optimization of the accessible information, is that the POVM in theorem 116 is the optimal only if the representation of the group G is irreducible.

5.2 Quantum states discrimination

In this section we discuss the minimal error to distinguish the states in the ensemble. We define the optimization problem of the minimal average probability of error to distinguish the states. We also present a POVM named pretty good measurement, which is optimal for G-covariant ensembles. We will define the the dephased-POVM, which is a dephasing map acting on the elements of the POVM. The dephased-POVM represents a projective measurement in the minimal error problem. Finally we compare the probability of success to distinguish the states via POVM and the analogous dephased-POVM.

Suppose Alice will send a fix number of states $\{p_x, \rho_x\}_{x=0,\dots,M-1}$ to Bob and he has to apply the best strategy to distinguish the states. The *minimal error probability* consists in Bob minimizing the average probability of error.

Definition 121. *Given an ensemble $\xi = \{p_x, \rho_x\}_{x=0,\dots,M-1}$, the minimal error to distinguish the states in the ensemble is defined as:*

$$P_{err} = \min_{\{A_k\}\subset POVM} \sum_{l=0}^{M-1} \sum_{k\neq l=0}^{M-1} p_l Tr[A_k \rho_l], \tag{5.96}$$

where the optimization is taken over POVMs with elements $\{A_k\}_{k=0,\dots,M-1}$ and the number of elements in the POVM is equal to the elements in the ensemble.

As the probability of success is $P_{suc} = 1 - P_{err}$ by definition it will be:

$$P_{suc} = \min_{\{A_k\}\subset POVM} \sum_k p_k Tr[A_k \rho_k]. \tag{5.97}$$

On the other hand solv ing this problem analytically is not easy and just some cases are known. Am interesting class of measurements which is not optimal in general but is always almost optimal is named *pretty good measurement* (PGM) [74]. This class of POVM just depends on the ensemble. It can be easily constructed for ensembles of pure states. Suppose an ensemble of states $\xi = \{p_k, \rho_k\}_{k=0,\dots,M-1}$ and a set of rank-1 operators $M_l = |\mu_l\rangle\langle\mu_l|$, we define a matrix with elements $P_{ij} = \sqrt{p_j}\langle\mu_i|\psi_j\rangle$. For the set of operators M_l be a valid measurement the square of P must be a Gram matrix G, in other words it must satisfy $(P^\dagger P)_{ij} = G_{ij}$, where $\sum_i G_{ii} = 1$ and $G_{ij} = G_{ji}$:

$$(P^\dagger P)_{ij} = \sum_k \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\mu_k\rangle\langle\mu_k|\psi_j\rangle = \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\psi_j\rangle = G_{ij}. \tag{5.98}$$

An artless way to produce a vector P which always satisfies this condition is assuming $(P^2)_{ij} = G_{ij}$. Then we must find a set of vectors $\{\mu_k\}$ which satisfies this condition:

$$\begin{aligned}(P^2)_{ij} &= G_{ij} = \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\psi_j\rangle = \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\rho^{-1/2}\rho\rho^{-1/2}|\psi_j\rangle, & (5.99)\\ &= \sum_k \sqrt{p_i}\sqrt{p_j}p_k\langle\psi_i|\rho^{-1/2}|\psi_k\rangle\langle\psi_k|\rho^{-1/2}|\psi_j\rangle, & (5.100)\end{aligned}$$

where in the Eq.5.100 we substituted $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, therefore the elements $P_{ij} = \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\rho^{-1/2}|\psi_j\rangle$ satisfy the condition for a POVM which elements $M_l = |\mu_l\rangle\langle\mu_l|$ where the vectors $|\mu_l\rangle = \sqrt{p_l}\rho^{-1/2}|\psi_l\rangle$

and $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. This POVM defines a measurement whose elements are almost orthogonal to the states in the ensemble [1] and it is named *pretty good measurement* [74]. For pure state, it is easy to compute the probability of success via PGM:

$$P_{suc}^{PGM} = \sum_k p_k \langle\psi_k| \rho^{-1/2} |\psi_k\rangle\langle\psi_k| \rho^{-1/2} |\psi_k\rangle, \tag{5.101}$$

substituting $P_{ii} = p_i \langle\psi_i| \rho^{-1/2} |\psi_i\rangle$ we have:

$$P_{suc}^{PGM} = \sum_k (P)_{kk}^2 = \sum_k (\sqrt{G})_{kk}^2. \tag{5.102}$$

The relation between the probability of error to distinguish the states via PGM in the minimum probability of error is proved in the following theorem:

Definition 122 (Pretty good measurement)**.** *Given an ensemble of states $\xi = \{p_x, \rho_x\}_{x=0,\dots,M-1}$, where $\rho_x \in \mathcal{D}(\mathbb{C}^d)$, the pretty good measurement (PGM) is defined as a POVM $\mathcal{M}^{PGM}$ with elements:*

$$M_k^{PGM} = p_k \rho^{-1/2} \rho_k \rho^{-1/2}, \tag{5.103}$$

where $\rho = \sum_{k=0}^{M-1} p_k \rho_k$ and the inverse is the Moore–Penrose pseudoinverse, which is taken over the non-zero eigenvalues of ρ.

Therefore we can calculate the probability of error to distinguish the states instead of all possible measurement but applying the PGM. This POVM is named pretty good because it performs reasonable well for any ensemble of states.

Theorem 123 ([12])**.** *Given an ensemble $\xi = \{p_x, \rho_x\}_{x=0,\dots,M-1}$, where $\rho_x \in \mathcal{D}(\mathbb{C}^d)$, the probability of error applying PGM is upper bounded by the optimal probability of error as:*

$$P_{err}^{PGM} \leq 2P_{err}, \tag{5.104}$$

where P_{err} is define in Eq.5.96.

Proof. Considering a state $\sigma = \sum_{i=0}^{M-1} p_i |i\rangle\langle i|$, where the states $\{|i\rangle\}_{i=0,\dots,M-1}$ form a orthonormal base on $\mathbb{C}^M$, where $\dim(\mathbb{C}^M) = M$. We also define a quantum channel $\mathcal{A} \in \mathcal{C}(\mathbb{C}^M, \mathbb{C}^d)$, such that $\mathcal{A}(|i\rangle\langle i|) = \rho_x$ and a measurement map $\mathcal{R} \in \mathcal{P}(\mathbb{C}^d, \mathbb{C}^M)$, such that $\mathcal{R}(\rho_x) = \sum_y Tr[E_y \rho_x] |y\rangle\langle y|$ and $\sum_y E_y = \mathbb{I}_d$. Finally we define the *classical fidelity*: $F_{class}(\sigma, \mathcal{M})$ for a state σ and a channel $\mathcal{M} \in \mathcal{P}(\mathbb{C}^M, \mathbb{C}^M)$, with elements $\{M_l\}_{l=0}^{M-1}$ as:

$$F_{class}(\sigma, \mathcal{M}) = \sum_i p_i \langle i| \mathcal{M}(|i\rangle\langle i|) |i\rangle = \sum_i p_i \sum_l |Tr(M_l |i\rangle\langle i|)|^2. \tag{5.105}$$

Before proving the theorem we must prove two lemmas about the classical fidelity:

Lemma 124. *Given an ensemble $\xi = \{p_x, \rho_x\}_{x=0,\dots,M-1}$ and defining the probability of success to distinguish the states on ξ via the POVM $\mathcal{R} = \{E_l\}$ as $P_{suc}(\xi, \mathcal{R}) = \sum_l p_l Tr[\rho_l E_l]$, we have:*

$$F_{class}(\sigma, \mathcal{R} \circ \mathcal{A}) = P_{suc}(\xi, \mathcal{R}). \tag{5.106}$$

Proof of Lemma.124.

$$F_{class}(\sigma, \mathcal{R} \circ \mathcal{A}) = \sum_i \langle i| \mathcal{R}[\mathcal{A}(|i\rangle\langle i|)] |i\rangle = \sum_i \langle i| \mathcal{R}[\rho_i] |i\rangle = \sum_i p_i Tr[E_i \rho_i] = P_{suc}(\xi, \mathcal{R}). \tag{5.107}$$

□

Lemma 125. *For a map $\mathcal{A} = \{A_{ij} = \sqrt{\lambda_{ij} |v_{ij}\rangle\langle i|}\}$ and $\mathcal{R}_{\sigma,\mathcal{A}} = \{R_i = \sigma^{1/2} A_i \mathcal{A}(\sigma)^{-1/2}\}$ the classical Fidelity represents the probability of success to distinguish the states in ξ via the pretty good measurement for ξ.*

Proof of Lemma.125.

$$F_{class}(\sigma,\mathcal{R}_{\sigma,\mathcal{A}}\circ\mathcal{A}) = \sum_i p_i \sum_{l,n}\sum_{k,j} |Tr[\sigma^{1/2}A_l A(\sigma)^{-1/2}A_j^k |i\rangle\langle i|]|^2, \tag{5.108}$$

although $\sigma^{1/2}|i\rangle = \sqrt{p_i}|i\rangle$ and $A_k^j|i\rangle\langle i| A_n^l = \delta_{ji}\delta_{il}\sqrt{\lambda_{kj}}\sqrt{\lambda_{ln}}|v_{kj}\rangle\langle v_{ln}|$, thus for $\mathcal{A}(\sigma) = \rho = \sum_i p_i\rho_i$:

$$\begin{aligned}
F_{class}(\sigma,\mathcal{R}_{\sigma,\mathcal{A}}\circ\mathcal{A}) &= \sum_i p_i \sum_{l,k} p_i\lambda_{ki}\lambda_{li}|Tr[\rho^{-1/2}|v_{ki}\rangle\langle v_{li}|]|^2 &(5.109)\\
&= \sum_i p_i^2 \sum_{l,k}\lambda_{ki}\lambda_{li}\langle v_{li}|\rho^{-1/2}|v_{ki}\rangle\langle v_{ki}|\rho^{-1/2}|v_{li}\rangle &(5.110)\\
&= \sum_i p_i^2\sum_l \lambda_{li}\langle v_{li}|\rho^{-1/2}\rho_i\rho^{-1/2}|v_{li}\rangle &(5.111)\\
&= \sum_i p_i Tr[p_i\rho_i\rho^{-1/2}\rho_i\rho^{-1/2}] &(5.112)\\
&= P_{suc}^{PGM}. &(5.113)
\end{aligned}$$

□

Now we can use these lemmas to prove the theorem. Given the classical fidelity for any measurement map, we named the probability of success as P_{succ}^{any}, by Lemma.124:

$$P_{suc}^{any} = F_{class}(\sigma,\mathcal{R}\circ\mathcal{A}), \tag{5.114}$$

where the map $\mathcal{R} = \{R_i\}$, and without loss of generality we can assume that R_i act in the $supp(\mathcal{A}(\sigma))$, therefore there is an operator B_i such that $R_i = \sigma^{1/2}B_i\mathcal{A}(\sigma)^{-1/2}$, then:

$$P_{suc}^{any} = F_{class}(\sigma,\mathcal{R}\circ\mathcal{A}) = \sum_l p_l\sum_{i,j}|Tr[\sigma^{1/2}B_i\mathcal{A}(\sigma)^{-1/2}A_j|l\rangle\langle l|]|^2. \tag{5.115}$$

Define:

$$X_{ij}^l := Tr[\sigma^{1/2}B_i^\dagger\mathcal{A}(\sigma)^{-1/2}A_j|l\rangle\langle l|], \tag{5.116}$$

getting a good choice of $\{B_i^l\}$ and $\{A_i^l\}$ such that they correspond to the singular decomposition of the matrix X^l:

$$Y_i i^l = Tr[\sigma^{1/2}B_i^\dagger\mathcal{A}(\sigma)^{-1/2}A_i|l\rangle\langle l|], \tag{5.117}$$

therefore:

$$P_{suc}^{any} = \sum_l p_l\sum_i|Tr[\sigma^{1/2}B_i\mathcal{A}(\sigma)^{-1/2}A_i^l|l\rangle\langle l|]|^2. \tag{5.118}$$

We can rewrite $Tr[\sigma^{1/2}B_i\mathcal{A}(\sigma)^{-1/2}A_i^l|l\rangle\langle l|] = Tr[|l\rangle\langle l|\sigma^{1/4}B_i\mathcal{A}(\sigma)^{-1/4}\mathcal{A}(\sigma)^{-1/4}A_i^l\sigma^{1/4}|l\rangle\langle l|]$, then defining:

$$\begin{aligned}
X_{il} &:= \mathcal{A}(\sigma)^{-1/4}A_i^l\sigma^{1/4}|l\rangle\langle l|; &(5.119)\\
Y_{il} &:= \mathcal{A}(\sigma)^{-1/4}B_i^l\sigma^{1/4}|l\rangle\langle l|, &(5.120)
\end{aligned}$$

substituting in Eq.5.118:

$$\begin{aligned}
P_{suc}^{any} &= \sum_l p_l\sum_i|Tr[Y_{il}^\dagger X_{il}]|^2, &(5.121)\\
&\leq \sum_l p_l\sum_i|Tr[Y_{il}^\dagger Y_{il}]Tr[X_{il}^\dagger X_{il}]|, &(5.122)\\
&\leq \left\{\sum_l p_l\sum_i|Tr[Y_{il}^\dagger Y_{il}]|^2\sum_k|Tr[X_{kl}^\dagger X_{kl}]|^2\right\}^{1/2}, &(5.123)
\end{aligned}$$

where in the first and second inequalities we used the triangle inequality. As the sum in Y_{il} is smaller than 1:

$$\sum_i |Tr[\sigma^{1/2}B_i^\dagger \mathcal{A}(\sigma)^{-1/2}B_i^l \,|l\rangle\langle l|]|^2 \leq \sum_{ij} |Tr[\sigma^{1/2}B_i^\dagger \mathcal{A}(\sigma)^{-1/2}B_{jk}^l \,|l\rangle\langle l|]|^2 = F_{class}(\sigma, \mathcal{R}\circ\mathcal{B}) \leq 1.$$

Substituting this bound in Eq.5.123:

$$\begin{aligned} P_{suc}^{any} &\leq \left\{\sum_l p_l \sum_k |Tr[X_{kl}^\dagger X_{kl}]|^2\right\}^{1/2}, &(5.124)\\ &\leq \left\{\sum_l p_l \sum_{kj} |Tr[X_{kl}^\dagger X_{jl}]|^2\right\}^{1/2}, &(5.125)\\ &= \sum_i |Tr[\sigma^{1/2}\mathcal{A}(\sigma)^{-1/2}A_i^l \,|l\rangle\langle l|]|^2, &(5.126)\\ &= F_{class}(\sigma, \mathcal{R}_{\sigma,\mathcal{A}}\circ\mathcal{A}), &(5.127)\\ &= (P_{suc}^{PGM})^{1/2}, &(5.128)\end{aligned}$$

where in the second inequality we just summed another M terms. As $P_{suc} = 1 - P_{err}$ we have:

$$1 - P_{err}^{PGM} \geq (1 - P_{err}^{any})^2, \tag{5.129}$$

or just

$$P_{err}^{PGM} \leq 2P_{err}^{any} - (P_{err}^{any})^2 \leq 2P_{err}^{any}. \tag{5.130}$$

□

As the main goal of this chapter is to compare general measurements with projective measurements we must obtain an analogous of the probability of success restricted to projective measurements. Although we cannot just substitute projective measurements in the min-error expression Eq.5.96, as the interesting case is when the number of states in the ensemble is greater than the dimension of the states in the ensemble. If the number of states in the ensemble is less than the dimension, even for linearly dependent states, projective measurement are enough [1]. Therefore when the number of states in the ensemble is greater than the dimension we need a POVM with more elements than the dimension in the calculus of the minimal error probability. However we know that a POVM whose all the elements commute is just a projective measurement. Thus for a given POVM with elements $\{M_k\}_{k=0,\ldots,M-1}$ we define a dephased-POVM with elements $\{M_l^\Pi = \Pi(M_l)\}_{l=0,\ldots,M-1}$, where $\Pi(A) = \sum_\beta |e_\beta\rangle\langle e_\beta| A |e_\beta\rangle\langle e_\beta|$ is a dephasing channel acting on the operator A. If the POVM is composed by rank-1 operators, the dephased-POVM will be composed by operators with rank=dim$\{\text{supp}(\Pi(M_l))\}$.

Definition 126. *For a given POVM $\mathcal{M}$ with elements $\{M_l\}_l$ and an ensemble $\xi = \{p_k, \rho_k\}_k$, the probability of success via the dephased-POVM optimizing over all dephasing channels is:*

$$P_{suc}^{\Pi} = \max_{\Pi} \sum_k p_k Tr[\Pi(M_k)\rho_k]. \tag{5.131}$$

As the dephasing channel is a linear CPTP map, the probability of success can be viewed as the best probability to distinguish quantum states when all of them are submitted to the same classical noise:

$$P_{suc}^{\Pi} = \max_{\Pi} \sum_k p_k Tr[M_k \Pi(\rho_k)]. \tag{5.132}$$

This property of CPTP maps was proved in Section.2. Given this property of CPTP maps we can calculate an upper bound for the difference in the probabilities of success via a POVM and the corresponding dephased-POVM which just depends of the ensemble.

Theorem 127. *The difference between the optimal probability of success P_{suc}^{opt} and the dephased-probability of success P_{suc}^{Π} for a ensemble $\xi = \{p_k, \rho_k\}_{k=0,\dots,M-1}$, where $\rho_k \in \mathcal{D}(\mathbb{C}^d)$, is upper bounded by the local-disturbance of the state: $\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \in \mathcal{D}(\mathbb{C}^M \otimes \mathbb{C}^d)$.*

$$P_{suc}^{opt} - P_{suc}^{\Pi} \leq \min_{\Pi} ||\rho_{XB} - \Pi(\rho_{XB})||_1. \tag{5.133}$$

Proof. If it is valid for any POVM, it will be valid for the optimal too, then given a POVM $\mathcal{M}$ with elements $\{M_l\}_l$:

$$P_{suc}^{\mathcal{M}} - P_{suc}^{\Pi} = \min_{\Pi} \sum_l p_l \{Tr[M_l \rho_l] - Tr[M_l \Pi(\rho_l)]\} = \min_{\Pi} \sum_l p_l Tr[M_l \{\rho_l - \Pi(\rho_l)\}], \tag{5.134}$$

given the Holder inequality $Tr[AB^\dagger] \leq ||A||_\infty ||B||_1$ for two operators A and B:

$$P_{suc}^{\mathcal{M}} - P_{suc}^{\Pi} \leq \min_{\Pi} \sum_l p_l ||M_l||_\infty ||\rho_l - \Pi(\rho_l)||_1, \tag{5.135}$$

as the matrices $\{M_l\}_l$ are elements of a POVM, they are positive and have trace less than one, hence $||M_l||_\infty \leq 1$, which implies:

$$P_{suc}^{\mathcal{M}} - P_{suc}^{\Pi} \leq \min_{\Pi} \sum_l p_l ||\rho_l - \Pi(\rho_l)||_1. \tag{5.136}$$

However the local-disturbance $D_1(\rho_{XB})$ of a classical-quantum state measured via trace-norm is:

$$D_1(\rho_{XB}) = \min_{\Pi} || \sum_x p_x |x\rangle\langle x| \otimes \rho_x - \sum_x p_x |x\rangle\langle x| \otimes \Pi(\rho_x)||_1 = \min_{\Pi} \sum_x p_x || \, |x\rangle\langle x| \, ||_1 ||\rho_x - \Pi(\rho_x)||_1, \tag{5.137}$$

as $|| \, |x\rangle\langle x| \, ||_1 = 1$ this proves the theorem because if this is valid for any POVM will be valid for the optimal. □

We can immediately realize that this bound is tight once that for an ensemble of orthogonal states the difference on the probabilities of success via POVM and dephased-POVM is zero. For orthogonal states the bipartite state ρ_{XB} is a classical-classical state, therefore the local-disturbance is also zero. Actually for classical-quantum states the local disturbance will be equal to the 1-norm geometrical quantum discord for classical-classical states. As we discussed in Sec.3.2.3 the 1-norm geometrical quantum discord for classical-classical states is equal to the local disturbance when the measurement is performed on both systems. Therefore as ρ_{XB} in Eq.5.133 is a classical-quantum states, it persists unchanged under the action of a measurement on the classical part.

Definition 128. *Given a POVM map $\mathcal{M} \in \mathcal{P}(\mathbb{C}^d, \mathbb{C}^M)$, with elements $\{M_l\}_{l=0,\dots,M-1}$ we define the degree of projectiveness of the POVM as:*

$$\Pi(\mathcal{M}) = \min_{\Pi} \sum_l ||M_l - \Pi(M_l)||_1, \tag{5.138}$$

where $\Pi \in \mathcal{P}(\mathbb{C}^d, \mathbb{C}^d)$ is a dephasing map.

This expression is optimal for the basis where the elements of the POVM can be written, such that the POVM map is the closest to a projective measurement. The degree of projectiveness will be zero when all of the elements in the POVM commute, in this way it can be viewed as a projective measurement.

For the pretty good measurement the degree of projectiveness can be limited by the difference of the probability of success for the PGM and the probability of success for the dephased PGM.

Proposition 129. *For an ensemble $\xi = \{p_k, \rho_k\}_{k=0,\dots,M-1}$ the degree of projectiveness of the PGM measurement with elements $M_l^{PGM} = p_l \rho^{-1/2} \rho_l \rho^{-1/2}$ is is limited above by the difference between the proba-*

bility of success via PGM and dephased-PGM:

$$\min_{\Pi} \sum_k ||M_k - \Pi(M_k)||_1 \geq P_{suc}^{PGM} - P_{suc}^{\Pi-PGM}, \tag{5.139}$$

where P_{suc}^{PGM} *and* $P_{suc}^{\Pi-PGM}$ *are the probability of success via PGM and the probability of success via the dephased-PGM.*

Proof.

$$P_{suc}^{PGM} - P_{suc}^{\Pi-PGM} = \min_{\Pi} \sum_l p_l \{Tr[M_l \rho_l] - Tr[\Pi(M_l)\rho_l]\} = \min_{\Pi} \sum_l p_l Tr[\rho_l \{M_l - \Pi(M_l)\}], \tag{5.140}$$

again applying the Holder inequality :

$$P_{suc}^{PGM} - P_{suc}^{\Pi-PGM} \leq \min_{\Pi} \sum_l p_l ||\rho_l||_\infty ||M_l - \Pi(M_l)||_1, \tag{5.141}$$

and $p_l ||\rho_l||_\infty \leq 1$:

$$P_{suc}^{PGM} - P_{suc}^{\Pi-PGM} \leq \min_{\Pi} \sum_l ||M_l - \Pi(M_l)||_1. \tag{5.142}$$

□

For a G-covariant ensemble of pure states $\xi_{|G|} = \{p_k, |\psi_k\rangle\langle\psi_k|\}_{k=0,\ldots,M-1}$ it is known that the PGM is the optimal measurement [10]. The elements of the POVM are proportional to the states of the ensemble, because the convex combination of the states in the ensemble $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| = \mathbb{I}_d$. By the definition of the PGM, the elements of the POVM are [10]:

$$M_k^{PGM} = p_k \rho^{-1/2} |\psi_k\rangle\langle\psi_k| \rho^{-1/2} = d p_k |\psi_k\rangle\langle\psi_k|, \tag{5.143}$$

where $|\psi_k\rangle \in \mathbb{C}^d$. For an ensemble whose the states are linearly independent, the PGM is also the optimal measurement, and it is a projective measurement [111]. Then if the number of the elements in the ensemble is smaller than the dimension, for G-covariant ensembles, the optimal measurement is a projective measurement. Given Eq.5.143 we can calculate the probability of success to discriminate the state of a G-covariant ensemble, as the PGM is the optimal measurement, it is given by:

$$P_{suc}^{PGM} = \sum p_k \mathrm{Tr}[M_k^{PGM} |\psi_k\rangle\langle\psi_k|] = d \sum_k p_k^2. \tag{5.144}$$

If we consider a ensemble of equiprobable states $\xi_G = \xi_{|G|} = \{1/M, |\psi_k\rangle\langle\psi_k|\}_{k=0}^{M-1}$ the probability of success is simply:

$$P_{suc}^{PGM} = \frac{d}{M}. \tag{5.145}$$

It is not simple calculate a general expression for the probability of success restricted to dephased-PGM. However we can calculated for a given G-covariant ensemble. Then we define the 2-design ensemble [109].

Definition 130. *Consider an ensemble of equiprobable states* $\xi = \{1/n, P_k\}_{k=1}^n$*, where* $P_k \in \mathbb{C}^d$ *are rank-1 projectors, and* $d < n$*. The projectors* P_k *satisfy the following conditions:*

$$\tfrac{1}{n} \textstyle\sum_k P_k = \tfrac{\mathbb{I}}{d}; \tag{5.146}$$

$$\tfrac{1}{n} \textstyle\sum_k P_k \otimes P_k = \tfrac{1}{d(d+1)} (\mathbb{I} + \mathbb{F}), \tag{5.147}$$

where $\mathbb{F} = \sum_{i,j} |i,j\rangle\langle j,i|$ *is named Swap operator, and it has the following property* $\mathbb{F} = (U \otimes U)\mathbb{F}(U^\dagger \otimes U^\dagger)$*, and d is the dimension of the Hilbert space.*

Given the definition of the probability of success to discriminate the states in the ensemble defined above (Defi.130), when the measurement is restricted to the dephased-PGM (Eq.5.131):

$$P_{suc}^{\Pi-PGM} = \max_{\Pi} \sum_k \frac{1}{n} \text{Tr}(P_k \Pi(M_k)). \tag{5.148}$$

The two design ensemble is a G-covariant ensemble, once that there exist a irreducible unitary representation $\{U_k\}_k$ of a group G such that:

$$\frac{1}{n} \sum_k U_k P_0 U_k^\dagger = \frac{\mathbb{I}}{d}, \tag{5.149}$$

where P_0 is a specific projector such that $\{U_k P_0 U_k^\dagger\}_k$ satisfies the condition in Eq.5.147. If the projector P_0 is a special state named Fiducial state and the group G is the Weyl-Schwinger group, the set of state which satisfy the definition 130 is named spherical 2-design, with the minimum number of elements $n = d^2$, and the PGM for this ensemble is named SIC-POVM [152]. Therefore the POVM which optimizes the probability of success to discriminate the states in a 2-design ensemble of states is the PGM. Specially for this ensemble we can calculate the probability of success in Eq.5.148. It is interesting because this ensemble is symmetric enough to the probability of success to discriminate the states via a dephased POVM is independent on the dephasing. This statement is proved in the following theorem.

Theorem 131. *The probability of success to discriminate the states, in a* 2*-design ensemble, via a dephased-PGM is:*

$$P_{suc}^{\Pi-PGM} = \frac{2d}{n(d+1)}, \tag{5.150}$$

where n is the number of states in the ensemble, d is the dimension of the Hilbert space and $n > d$.

Proof. For a given dephasing Π applied on the PGM $\mathcal{M}^{PGM} = \{M_k = \frac{d}{n} P_k\}$:

$$P_{suc}^{\Pi-PGM} = \frac{d}{n^2} \sum_k \text{Tr}(P_k \Pi(P_k)), \tag{5.151}$$

as:

$$\begin{aligned} \text{Tr}(P_k \Pi(P_k)) &= \text{Tr}[\Pi(P_k)\Pi(P_k)] &(5.152)\\ &= \text{Tr}[\Pi(P_k) \otimes \Pi(P_k)\mathbb{F}] &(5.153)\\ &= \text{Tr}[\Pi \otimes \Pi(P_k \otimes P_k)\mathbb{F}]. &(5.154) \end{aligned}$$

The property in Eq.5.153 comes from the definition of the Swap operator: $\mathbb{F}(A \otimes B) = A \cdot B \otimes \mathbb{I}$. Substituting Eq.5.154 in Eq.5.151 we have:

$$\begin{aligned} P_{suc}^{\Pi-PGM} &= \frac{d}{n^2} \sum_k \text{Tr}[\Pi \otimes \Pi(P_k \otimes P_k)\mathbb{F}] &(5.155)\\ &= \frac{d}{n^2} \text{Tr}[\Pi \otimes \Pi(\sum_k P_k \otimes P_k)\mathbb{F}] &(5.156)\\ &= \frac{d}{n} \text{Tr}\left\{\Pi \otimes \Pi[\frac{1}{d(d+1)}(\mathbb{I} + \mathbb{F})]\mathbb{F}\right\} &(5.157)\\ &= \frac{d}{n} \frac{1}{d(d+1)} \left\{\text{Tr}[\Pi \otimes \Pi(\mathbb{I})\mathbb{F}] + \text{Tr}[\Pi \otimes \Pi(\mathbb{F})\mathbb{F}]\right\}, &(5.158) \end{aligned}$$

where in Eq.5.156 we used the linearity of the dephasing channel, and in Eq.5.157 we used the

Eq.5.147. Then, calculating each term separately we have:

$$\mathrm{Tr}[\Pi \otimes \Pi(\mathbb{I})\mathbb{F}] = \mathrm{Tr}[\mathbb{F}] = \mathrm{Tr}[\sum_{i,j} |i,j\rangle\langle j,i|] = d. \tag{5.159}$$

As the Swap operator is invariant under the action of local unitary operations, and the dephasing in Eq.5.158 is applied locally, the second term will be:

$$\mathrm{Tr}[\Pi \otimes \Pi(\mathbb{F})\mathbb{F}] = \sum_{k,l} \langle k,l| \mathbb{F} |k,l\rangle \langle k,l| \mathbb{F} |k,l\rangle \,, \tag{5.160}$$

for any basis $\{|k,l\rangle\}_{k,l}$, the expectation value of $\mathbb{F}$ is:

$$\langle k,l| \mathbb{F} |k,l\rangle = \delta_{k,l}. \tag{5.161}$$

Therefore the Eq.5.160 will be:

$$\mathrm{Tr}[\Pi \otimes \Pi(\mathbb{F})\mathbb{F}] = \sum_{k,l=1}^{d} \delta_{k,l} = d. \tag{5.162}$$

Hence, substituting Eq.5.159 and Eq.5.162 in Eq.5.158, we have:

$$P_{suc}^{\Pi-PGM} = \frac{2d}{n(d+1)}. \tag{5.163}$$

□

From this theorem we can obtain the difference between the the optimal probability of success to discriminate the states in the 2-design ensemble (Eq.5.145) and the probability of success restricted to projective measurements (Eq.5.163):

$$P_{suc}^{PGM} - P_{suc}^{\Pi-PGM} = \frac{d}{n}\left(\frac{d-1}{d+1}\right). \tag{5.164}$$

We can realize that the difference in above equation decreases with the increase of the number of the states in the ensemble, fixed the dimension. It means that, when the number of states in the ensemble is very large, perform the PGM or a projective measurement the ability to distinguish the states is very bad independent of the measurement.

5.3 Quantumness of correlations

In this section we shall discuss about the consequence of the embedding in the context of quantumness of correlations. We will study the quantum discord and compare two definitions of it, one definition which the optimization is taken over general measurements and the other restricted to projective measurements. We will use the results obtained in the last section to calculate a lower bound for the difference between the definitions for a class of states. We use the discussion of the last section to understand about the requirement of the optimization of the quantum discord over POVM for classical-quantum states.

5.3.1 Quantum discord: POVM vs Projective measurement

When quantum discord was introduced By H. Olivier and W. Zurek [115], the optimization was just over projective measurements, however in another seminal paper, L. Henderson and V. Vedral [76] introduced the optimization of quantum discord over general measurements (POVM). There is no problem in defining quantum discord over projective measurements because it can be easily extended to POVMs via Naimark's theorem, which states that a general measurement can be performed by projective measurement in a extended space. In this section we shall discuss about the consequence to restrict the optimization of quantum discord to projective measurements in comparison with POVMs.

The optimization of quantum discord over POVM is intractable. Therefore analytical calculation for quantum discord only can be performed for some specific states [6, 67, 100, 101]. Then we are interested in understand what are the consequences of the extension of the space in the optimization of quantum discord.

Given a state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, a local measurement map $\mathbb{I}_A \otimes \mathcal{M}_B \in \mathcal{P}(\mathbb{C}_{AB}, \mathbb{C}_{AY})$ and a projective measurement map $\mathbb{I}_A \otimes \Pi_B \in \mathcal{P}(\mathbb{C}_{AB}, \mathbb{C}_{AB})$, these maps act on ρ_{AB} as:

$$\mathbb{I}_A \otimes \mathcal{M}_B(\rho_{AB}) = \sum_{k=0}^{N-1} Tr_B[\mathbb{I}_A \otimes M_k^B \rho_{AB}] \, |k\rangle\langle k|_Y , \tag{5.165}$$

$$\mathbb{I}_A \otimes \Pi_B(\rho_{AB}) = \sum_{l=0}^{d_B-1} Tr_B[\mathbb{I}_A \otimes \Pi_l^B \rho_{AB}] \, |l\rangle\langle l|_B \tag{5.166}$$

where $\dim(\mathbb{C}_Y) = N$.

Definition 132. *We define the quantum discord and quantum discord restricted to projective measurement for a state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$:*

$$D(A:B)_{\rho_{AB}} = \min_{\mathbb{I}_A \otimes \mathcal{M}_B \in \mathcal{P}(\mathbb{C}_{AB}, \mathbb{C}_{AY})} \{ I(A:B)_{\rho_{AB}} - I(A:B)_{\mathbb{I}_A \otimes \mathcal{M}_B(\rho_{AB})} \} \tag{5.167}$$

$$D_{\Pi}(A:B)_{\rho_{AB}} = \min_{\mathbb{I}_A \otimes \Pi_B \in \mathcal{P}(\mathbb{C}_{AB}, \mathbb{C}_{AB})} \{ I(A:B)_{\rho_{AB}} - I(A:B)_{\mathbb{I}_A \otimes \Pi_B(\rho_{AB})} \}. \tag{5.168}$$

As we discussed an ensemble of states can be written as classical-quantum state which relates quantum discord with accessible information [53].

Proposition 133 ([53])**.** *The accessible information $I(\xi)$ for the ensemble ξ is equal to the classical correlation $J(X:B)_{\rho_{XB}}$ of the state ρ_{XB}.*

Proof. The definition of the measure of the classical correlations for a bipartite state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$, under measurement in B, is given by:

$$J(A:B)_{\rho_{AB}} = \max_{\mathcal{M} \in P(\mathbb{C}_B, \mathbb{C}_Y)} I(A:Y)_{\rho_{AY}}, \tag{5.169}$$

where $\rho_{AY} = \mathbb{I}_A \otimes \mathcal{M}_B(\rho_{AB}) = \sum_y Tr_B[\rho_{AB} \mathbb{I}_A \otimes M_B^y] \, |y\rangle\langle y|$. For $\rho_{AB} = \rho_{XB} = \sum_x p_x \, |x\rangle\langle x| \otimes \rho_x$ we have

$$J(X:B)_{\rho_{XB}} = \max_{\mathcal{M} \in P(\mathbb{C}_B, \mathbb{C}_Y)} I(X:Y)_{\rho_{XY}} = I(\xi), \tag{5.170}$$

where $\xi = \{p_i, \rho_i\}_{i=0,\ldots,M-1}$. □

Given the definition of quantum discord $D(A:B)_{\rho_{AB}} = I(A:B)_{\rho_{AB}} - J(A:B)_{\rho_{AB}}$, thus the quantum discord for a classical-quantum state $\rho_{XB} = \sum_x p_x \, |x\rangle\langle x| \otimes \rho_x$ can be written in function of the accessible information of the ensemble $\xi = \{p_x, \rho_x\}$:

$$D(X:B)_{\rho_{XB}} = I(X:B)_{\rho_{XB}} - I(\xi), \tag{5.171}$$

The mutual information for a classical-quantum state is equal the Holevo quantity for the ensemble ξ, in other words $I(X:B)_{\rho_{XB}} = \chi(\xi)$, then:

$$D(X:B)_{\rho_{XB}} = \chi(\xi) - I(\xi), \tag{5.172}$$

in this way the quantum discord for the classical-quantum state ρ_{XB} measures the amount of information which we cannot access from the ensemble ξ, once that the $\chi(\xi)$ quantifies the amount of information we can codify in the ensemble ξ in bits and the quantity $I(\xi)$ quantifies the amount of information we can extract in bits, performing the best decodification strategy. Therefore all the properties of the accessible information can be rescued for quantum discord of classical-quantum states. One interesting property which we will study in the next section is the additivity of quantum discord. It is known that for an ensemble composed by states in a 2-dimensional space, if the number of states in the ensemble is less or equal to the dimension, projective measurements are enough in the calculation of the accessible information [93]. For ensemble of states in a d-dimensional space it was conjectured that is also valid [93].

Given the expression Eq.5.172 we can calculate a lower bound for the difference of quantum discord and the quantum discord restricted to projective measurements for a class of classical-quantum states. An interesting class of classical-quantum states is the $G-$covariant states. For this kind of ensemble we know that the POVM which optimizes the quantum discord is also $G-$covariant. As we are applying the measurement in just one system we expect that we should not worry about the dimension of the system which we will not apply the measurement, however we can see that the dimension of the nonmeasured system is also important for the kind of measurement which will be applied. We can take the case of a classical-quantum state $\rho_{XB} \in \mathcal{D}(\mathbb{C}^M \otimes \mathbb{C}^2)$ which was studied in the last section, i.e.:

$$\rho_{XB} = \frac{1}{M}\sum_{x=0}^{M-1} |x\rangle\langle x| \otimes |\psi_k\rangle\langle\psi_k|\,, \tag{5.173}$$

where the states $|\psi_k\rangle = \cos \pi k/M\,|0\rangle + \sin \pi k/M\,|1\rangle$. For this case we calculate a lower bound of the difference between the optimal projective measurement and the optimal POVM in the accessible information for an ensemble composed by 2-dimensional states in Eq.**??**. Given Eq.**??** and Eq.5.172 we obtain a lower bound for the difference in the quantum discord and the quantum discord restricted to projective measurements for classical-quantum states, where the measurement is applied on a 2-dimensional system:

$$\begin{aligned} D(A:B)_{\rho_{XB}} - D_{\Pi}(A:B)_{\rho_{XB}} &= \frac{1}{M}\sum_k \{(\sin(\pi k/M))^2 \log[(\sin(\pi k/M))^2]\} - \\ &- \frac{1}{M}\sum_k \{(\cos(\pi k/M))^2 \log[(\cos(\pi k/M))^2]\}\,. \end{aligned} \tag{5.174}$$

We immediate realized that the measurement which optimize the quantum discord also depends on the system which is not being measured. For $2\otimes 2$ states it was studied numerically and realized that projective measurements are enough to calculate quantum discord [63]. However the same is not valid for $M\otimes 2$ systems even the measurement is performed on the 2-dimensional system. It can also be interesting to obtain a general upper bound for the difference in the optimization of quantum discord via POVM or projective measurements.

5.3.2 1-way work deficit under embedding

The 1-way work deficit of a state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_A \otimes \mathbb{C}_B)$ is a measure of the amount of pure states which can be distilled locally from ρ_{AB} in the limit of many copies under CLOCC (local unitary operations and one way classical communication). In the limit of one copy the 1-way work deficit measures the amount of quantum correlation in ρ_{AB}. Suppose now we couple a pure ancilla on ρ_{AB}. In the limit of many copies we can think this process as a catalysis, where pure states should be used to produce more pure states, what is not acceptable, therefore local pure ancillas must not increase the amount of pure states which can be extracted [79] [2]. In the limit of one copy, as coupling a pure ancilla cannot increase the amount of correlations, the 1-way work deficit must not change. Although this argument makes sense physically, a rigorous mathematical proof

[2] A mathematical proof of this argument can be find on Theorem 4 of [79].

still lacking. We are raising questions about the validity of this argument because mathematically coupling a pure ancilla and applying a projective measurement is the same of performing a POVM on the state. As we know that the projective measurement is a restricted kind of POVM, it is natural to think that the 1-way work deficit of the embedded state, in the limit of one copy, will be less than the original state.

Chapter 6

Quantum Correlations for indistinguishable particles

In this chapter we discuss the concept of quantum correlations for indistinguishable particles: bosonic or fermionic systems. We introduce the approach of entanglement witness for fermionic systems, and also the witnessed entanglement in this context . We obtain the Fermionic Generalized robustness [87]. We also discuss the concept of quantumness of correlations for indistinguishable particles. We shall obtain the set of classical correlated states in this context via the activation protocol. This set can be obtained from the absence of entanglement between the system and the measurement apparatus during the a local measuring. We also calculated from the distinguishable entanglement between the system and the measurement apparatus the relative entropy of quantumness for indistinguishable particles [86]. Given the definition of the set of states without quantumness of correlations we also define a geometrical discord based on trace distance [87].

6.1 The indistinguishable particle formalism

Indistinguishability is related to the nonexistence of intrinsic properties that allow us to identify the particles of the system individually. For classical systems, even if the particles are identical, it is always possible to label them and follow their paths in the phase space. Then we can identify them in any subsequent time. However for quantum systems, the kinematic objects are vectors in a complex vector space, from where only expectation values of the observables can be extracted. Then, there exists an intrinsic uncertainty about the position of the particles in each instant of time, which makes unfeasible the characterization of the paths [46].

Consider a system composed by two particles represented by the state:

$$|\phi\rangle = \sum_{n_1,n_2} c_{n_1,n_2} \left| n_1^{(1)} \right\rangle \left| n_2^{(2)} \right\rangle, \tag{6.1}$$

where $\left| n_x^{(y)} \right\rangle$ represents the particle x in the state y and $\{|n_x\rangle\}_{n=0,1,2,...}$ are orthogonal states in the space of particle x. We also can consider another state $|\phi'\rangle$:

$$|\phi'\rangle = \sum_{n_1,n_2} c_{n_1,n_2} \left| n_1^{(2)} \right\rangle \left| n_2^{(1)} \right\rangle. \tag{6.2}$$

The states $|\phi\rangle$ and $|\phi'\rangle$ are related by the permutation operator P_{12}:

$$P_{12} \left| n_1^{(1)} \right\rangle \left| n_2^{(2)} \right\rangle = \left| n_1^{(2)} \right\rangle \left| n_2^{(1)} \right\rangle, \tag{6.3}$$

hence $P_{12}|\phi\rangle = |\phi'\rangle$ and $P_{12}|\phi'\rangle = |\phi\rangle$. Therefore $P_{12} = P_{12}^{\dagger} = P_{12}^{-1}$. For the particles to be indistinguishable any observable g must satisfy:

$$\langle \phi | g | \phi \rangle = \langle \phi' | g | \phi' \rangle = \langle \phi | P_{12} g P_{12} | \phi \rangle , \tag{6.4}$$

then, any observable which acts on the space of the indistinguishable particles must commute with the permutation operator:

$$[P_{12}, g] = 0. \tag{6.5}$$

As the permutation operator must commute with any observable which acts on the system of indistinguishable particles, the identity operator can be decomposed as:

$$\mathbb{I} = \frac{\mathbb{I} - P_{12}}{2} + \frac{\mathbb{I} + P_{12}}{2} = \mathcal{A} \oplus \mathcal{S}. \tag{6.6}$$

Where $\mathcal{A}$ and $\mathcal{S}$ are named antisymmetric and symmetric operators, they define two subspaces in the space of linear operators. Given their definitions we can have:

$$\mathcal{A} = \mathcal{A}^\dagger = \mathcal{A}^2 \quad , \quad \mathcal{S} = \mathcal{S}^\dagger = \mathcal{S}^2 \quad \text{and} \quad \mathcal{A}\mathcal{S} = \mathcal{S}\mathcal{A} = 0. \tag{6.7}$$

As all observables commute with the permutation operator, considering an observable $g = g^\dagger$:

$$\mathcal{S} g \mathcal{A} = \mathcal{A} g \mathcal{S} = 0. \tag{6.8}$$

Hence, the observables cannot mixture the symmetric and antisymmetric subspaces. In this way a Hamiltonian, which describes the dynamic of the system of indistinguishable particles, must respect the symmetry of the state during the evolution. The same is valid for density matrices. A system of particles whose state is symmetric, with relation to the exchange of particles, cannot be transformed in an antisymmetric state. This impossibility to change a symmetric observable to an antisymmetric, and vice e versa, reflects intrinsic properties of the system of particles. If the state of the system is symmetric the particles of the system are named *Bosons*, if it is antisymmetric the particles are named *Fermions*. Applying the symmetric and antisymmetric operations on the state in eq.6.1 and normalizing, we have:

$$\mathcal{S} |\phi\rangle = |\phi_+\rangle = \frac{1}{\sqrt{2}} \sum_{n_1, n_2} c_{n_1, n_2} \left(\left| n_1^{(1)} \right\rangle \left| n_2^{(2)} \right\rangle + \left| n_1^{(2)} \right\rangle \left| n_2^{(1)} \right\rangle \right) \tag{6.9}$$

$$\mathcal{A} |\phi\rangle = |\phi_-\rangle = \frac{1}{\sqrt{2}} \sum_{n_1, n_2} c_{n_1, n_2} \left(\left| n_1^{(1)} \right\rangle \left| n_2^{(2)} \right\rangle - \left| n_1^{(2)} \right\rangle \left| n_2^{(1)} \right\rangle \right). \tag{6.10}$$

The antisymmetrization and symmetrization operations are also named *Slater permanent* and *Slater determinant* respectively. Antisymmetric states respect the Pauli exclusion principle, which states that fermions cannot be found in the same state. Therefore the probability amplitude of the particles be in the same state is zero, as we can check applying $\langle n_1 = 0, n_2 = 0|$ on $|\phi_-\rangle$: $\langle n_1 = 0, n_2 = 0 | \phi_- \rangle = 0$.

As discussed, the space state for indistinguishable fermions or bosons is antisymmetric or symmetric under permutation of particles. For these systems is convenient to use the *second quantization* formalism, in order to deal with the antisymmetric or symmetric states in the *Fock space*. Accordingly we introduce an algebra of operators which satisfies the following anti-commutation relations for systems of fermions, and commutation relation for bosons:

$$\{f_i^\dagger, f_j^\dagger\} = \{f_i, f_j\} = 0, \qquad \{f_i, f_j^\dagger\} = \delta_{ij}, \tag{6.11}$$

$$[b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0, \qquad [b_i, b_j^\dagger] = \delta_{i,j}, \tag{6.12}$$

where $f_i^\dagger$ and $g_i^\dagger$ are the fermionic and bosonic creation operators respectively, f_i and b_i are the fermionic and bosonic annihilation operators, respectively. The action of the annihilation operator on the vacuum state ($|vac\rangle$) is defined:

$$f_i |vac\rangle = 0, \tag{6.13}$$

$$g_i |vac\rangle = 0. \tag{6.14}$$

The vacuum state in the Fock space is defined as the absence of particles. We denote the Fock space of n fermions(bosons) sharing a d-dimensional single particle space as: $\mathcal{F}_n^d$ ($\mathcal{B}_n^d$).

An immediate consequence of the antisymmetric structure of the state space can be realized even in the simplest example of a two-fermion system, which, if analyzed in the usual way, will always be considered entangled. We must therefore rethink the way entanglement is calculated for systems of indistinguishable particles, as well as its physical interpretation.

In the case which the identical particles are localized in *distinct* laboratories and *independently* prepared, it is natural to think that the entanglement calculated in the usual way should not have any relevant physical meaning; or rather, "no quantum prediction, referring to an atom located in our laboratory, is affected by the mere presence of similar atoms in remote parts of universe" [124].

We are interested in the case of identical particles that are sufficiently close together such that the overlap between their wave functions is no longer negligible, and therefore they are indistinguishable. Fermionic systems of this kind can be described using Slater determinants. Consider, for example, a two-fermion state represented by a single Slater determinant, namely,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\phi\rangle) = f_\phi^\dagger f_\chi^\dagger |0\rangle, \tag{6.15}$$

where $|\phi\rangle$ and $|\chi\rangle$ correspond to orthonormal wave functions (*spin-orbitals*). It is easy to see, in this simple case, that the anti-symmetrization of coordinates introduces correlations between the fermions, namely, the well known *exchange contributions* from the Hartree-Fock theory. On the other hand, a single Slater determinant is solution of a one-particle Schrödinger equation and, therefore, can have no quantum correlation between the particles. Considering states described by more than one Slater determinant introduces additional correlations beyond the exchange contribution. We will then interpret such additional correlations as the analog of quantum entanglement in systems of distinguishable particles, calling them as *fermionic entanglement* [56].

A measure of fermionic entanglement was proposed as the analogous of Wootters concurrence [164]. Notwithstanding, such measure, called Schliemann concurrence (C_S), is valid only for two-fermion states with a four-dimensional single-particle Hilbert space ($\mathcal{F}_2^4$), i.e. the antisymmetric space of lowest dimension where can exist quantum correlated states. For distinguishable particles the lowest dimension where can exist correlations is $2 \otimes 2$, nonetheless for fermions the lowest dimension of a composed space is $4 \otimes 4$, because each particle has 2 possible spin configurations and 2 possible modes, and by the Pauli principle of exclusion they cannot be in the same mode with the same spin. For bosons, because the exclusion principle is not valid, the smallest composed system has dimension $2 \otimes 2$, which is two possible configurations of spin for each particle.

In order to define the Schliemann concurrence, we have to introduce some operators. Let $\mathcal{U}_{ph}$ be the operator of particle-hole transformation:

$$\mathcal{U}_{ph} f_i^\dagger \mathcal{U}_{ph}^\dagger = f_i, \qquad \mathcal{U}_{ph} |vec\rangle = \prod_{i=1}^{d} f_i^\dagger |vec\rangle, \tag{6.16}$$

being d the single-particle Hilbert space dimension. Similarly, define $\mathcal{K}$ as the anti-linear operator of complex-conjugation, satisfying the following relations:

$$\mathcal{K} f_i^\dagger \mathcal{K} = f_i^\dagger, \qquad \mathcal{K} f_i \mathcal{K} = f_i, \qquad \mathcal{K} |vec\rangle = |vec\rangle. \tag{6.17}$$

Thus, given the operator $\mathcal{D} = \mathcal{K}\mathcal{U}_{ph}$, called operator of dualisation, and the dual states $\widetilde{\rho} =$

$\mathcal{D}\rho\mathcal{D}^{-1}$, we have that the Schliemann concurrence for states $\rho \in \mathcal{D}(\mathcal{F}_2^4)$ is given by

$$C_S(\rho) = max(0, \lambda_6 - \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 - \lambda_1), \tag{6.18}$$

where $\lambda_i' s$ are, in descending order of magnitude, the square roots of the singular values of the matrix $R = \rho\tilde{\rho}$. The reader can find more details about the theory of entanglement for fermionic and bosonic systems in the review [56].

6.2 Entanglement Witness for Fermionic systems

The states without entanglement for indistinguishable particles are those that can be described by a single Slater determinant, or a convex mixture of them. Consider $\mathcal{F}_n^d$ as the Fock space of n indistinguishable fermions sharing a d-dimensional single-particle space. As introduced by Iemini *et al.* the following definition of "separable" states [87]:

Definition 134 (State with no fermionic entanglement (separable)). *A state $\sigma \in \mathcal{D}(\mathcal{F}_n^d)$ has no fermionic entanglement if it can be decomposed as*

$$\sigma = \sum_i p_i \; a_1^{i\dagger} \cdots a_n^{i\dagger} \left|vec\right\rangle \left\langle vec\right| a_n^i \cdots a_1^i, \quad \sum_i p_i = 1, \tag{6.19}$$

where $a_k^{i\dagger} = \sum_{l=1}^{d} c_l^{ik} f_l^\dagger$, and $\{f_l^\dagger\}$ is an orthonormal basis of fermionic creation operators for the space of a single fermion ($\mathcal{F}_1^d$).

The states defined by Eq.6.19 are not separable in the usual mathematical sense, meaning that they are product states or convex mixtures of it. But we will insist in referring to them as *separable*, for they are just anti-symmetrization of the usual distinguishable separable states. Entanglement, in the case of distinguishable particles, is defined in opposition to separability, i.e., an entangled state is that one which is not separable. We want to keep this notion.

It is interesting to note that, as in the case of distinguishable particles, the set of separable states is invariant under *local operations*, taking now into account that the local operations must be symmetric, due to the indistinguishability of the particles. Let Φ be a *local symmetric operation* (LSO), i.e., an operation that respects the Pauli exclusion principle and does not involve any interaction between particles. An LSO can be written as:

$$\Phi(\rho) = \sum_i (M_i \otimes M_i \otimes \cdots \otimes M_i)\, \rho\, (M_i^\dagger \otimes M_i^\dagger \otimes \cdots \otimes M_i^\dagger) \tag{6.20}$$

where M_i is a linear operator acting on the Hilbert space of a single particle. Given a fermionic separable pure state (i.e. a single Slater determinant) $\left|\psi_{sep}\right\rangle = \mathcal{A}(\left|e_1\right\rangle \otimes \cdots \otimes \left|e_n\right\rangle)$, where $\mathcal{A}$ is the anti-symmetrization operator, $\{\left|e_i\right\rangle\}$ is an orthonormal basis, and noting that $[\Phi, \mathcal{A}] = 0$, we see that

$$\begin{aligned}
(M_i \otimes \cdots \otimes M_i) \left|\psi_{Sep}\right\rangle &= (M_i \otimes \cdots \otimes M_i)\, \mathcal{A} \left|e_1 \cdots e_n\right\rangle \\
&= \mathcal{A}\, (M_i \otimes \cdots \otimes M_i) \left|e_1 \cdots e_n\right\rangle \\
&= \mathcal{A}\, \left|(M_i e_1) \cdots (M_i e_n)\right\rangle \\
&= \mathcal{A}\, \left|e_1' \cdots e_n'\right\rangle \\
&= \left|\psi_{Sep}'\right\rangle
\end{aligned} \tag{6.21}$$

and such result clearly extends to mixed states. Summarizing, given a separable state σ is separable, we have that $\Phi_{LSO}(\sigma)$ is also separable, indicating that in order to have quantum entanglement, the particles must interact by means of some global operation.

Now we adapt Brandão and Vianna's [27] technique in order to obtain a new algorithm to determine OEWs for indistinguishable fermions in the Fock space. The new method was proposed in [87], it can be enunciated as follows.

Theorem 135 (Determination of OEW using RSDP [87]). *A fermionic state $\rho \in \mathcal{D}(\mathcal{F}_n^d)$ is entangled if and only if the optimal value of the following RSDP is negative:* minimize $Tr(W\rho)$ subject to

$$\begin{array}{c} \sum_{i_{n-1}=1}^{d} \cdots \sum_{i_1=1}^{d} \sum_{j_1=1}^{d} \cdots \sum_{j_{n-1}=1}^{d} (c_{i_{n-1}}^{n-1^*} \cdots c_{i_1}^{1^*} c_{j_1}^{1} \cdots c_{j_{n-1}}^{n-1} \times \\ W_{i_{n-1}\cdots i_1 j_1 \cdots j_{n-1}}) \geq 0, \\ \forall c_i^k \in \mathcal{C},\ 1 \leq k \leq (n-1),\ 1 \leq i \leq d, \\ \mathcal{A} W \mathcal{A}^\dagger = W, \\ W \leq \mathcal{A}, \end{array} . \tag{6.22}$$

where d is the dimension of the single particle Hilbert space, $\{f_l^\dagger\}$ is an orthonormal basis of fermionic creation operators, $\mathcal{A}$ is the anti-symmetrization operator, and $\Phi(W) = W_{i_{n-1}\cdots i_1 j_1 \cdots j_{n-1}} = f_{i_{n-1}} \cdots f_{i_1} W f_{j_1}^\dagger \cdots f_{j_{n-1}}^\dagger \in \mathcal{P}(\mathcal{F}_1^d)$ is an operator acting on the space of one fermion.

The notation $W \leq \mathcal{A}$ means that $(\mathcal{A} - W) \geq 0$ is a positive semidefinite operator. If ρ is entangled, the operator W that minimizes the problem corresponds to the OEW of ρ.

Proof. It is known that a state is entangled if and only if there exists a witness operator W such that $Tr(W\rho) < 0$ and $Tr(W\sigma) \geq 0$ for every separable state σ. Consider a general separable state as given by Eq.6.19. The semi-positivity condition $Tr(W\sigma) \geq 0$ is equivalent to:

$$\langle 0|\, a_n a_{n-1} \cdots a_1\, W\, a_1^\dagger \cdots a_{n-1}^\dagger a_n^\dagger \,|0\rangle \geq 0, \tag{6.23}$$

for all $\{a_k^\dagger\}_k$ Note however that to satisfy Eq.6.23, it is sufficient that the operator $a_{n-1} \cdots a_1\, W\, a_1^\dagger \cdots a_{n-1}^\dagger$ is positive semidefinite. Thus follows directly that the operator W satisfying the problem in Eq.6.22 corresponds to an optimal entanglement witness. □

The RSDP given above is solved by means of probabilistic relaxations it terms of SDPs, as done in [27], where the set of infinite constraints is exchanged by a finite sample. Thus the witness operator obtained is such that satisfy most of the constraints in Eq.6.22. The small probability (ϵ) that a constraint be violated (i.e. $Tr(W\sigma) < 0$) diminishes as the size of the sample of constraints increases.

The constraint $\mathcal{A}W\mathcal{A}^\dagger = W$ restricts the operator to the space of antisymmetric entanglement witnesses ($\mathcal{W}(\mathcal{F}_n^d) = \mathcal{A}\mathcal{W}\mathcal{A}^\dagger$). The other constraint, $W \leq \mathcal{A}$, follows directly from the anti-symmetrization of the constraint of the entanglement witnesses of the generalized robustness, and implies that the OEW corresponds to the anti-symmetrized version of the Generalized Robustness, namely,

$$\mathcal{R}_g^{\mathcal{F}}(\rho) = max(0, -\min_{\mathcal{M}=\{W \in \mathcal{W}(\mathcal{F}_n^d)\,|\,W \leq \mathcal{A}\}} Tr(W\rho)). \tag{6.24}$$

$\mathcal{R}_g^{\mathcal{F}}(\rho)$ measures the minimum required mixture with a fermionic state such that all the entanglement of ρ is washed out. In other words, the Generalized Robustness is the minimum value of s such that

$$\sigma = \frac{\rho + s\varphi_f}{1+s} \tag{6.25}$$

be a separable state (Eq.6.19), where φ_f can be any fermionic state.

6.3 Quantumness of correlations in indistinguishable particles systems

As discussed above, for indistinguishable particles the space of quantum states is restricted to symmetric $\mathcal{S}$ or antisymmetric $\mathcal{A}$ subspaces, depending on the bosonic or fermionic nature of the system, and the particles are no longer accessible individually, thus eliminating the usual notions of separability and local measurements, and making the analysis of correlations much subtler. Note that the correlations between modes in a system of indistinguishable particles is subsumed in the usual analysis of correlations in systems of distinguishable ones. Thus we shall characterize and quantify a general notion of quantum correlations (not necessarily entanglement) genuinely arising between indistinguishable particles. We shall call these correlations by quantumness of correlations, to distinguish from entanglement, and it has an interpretation analogous to the quantumness of correlations in systems of distinguishable particles, as we shall see. One must however be careful with such phraseology, since systems of indistinguishable particles always have *exchange correlations* coming from the symmetric or antisymmetric nature of the wavefunction. The intrinsic exchange correlations are not included in the concept of the quantumness of correlations.

6.3.1 Activation protocol and the set of classical correlated states

As aforesaid, quantum correlations between distinguishable particles can be interpreted via a unavoidable entanglement created with the measurement apparatus in a partial von Neumann measurement on the particles, see Section 4.2. In systems of indistinguishable particles the notion of "local measurement" will be implemented through the algebra of single-particle observables (see for example Ref. [8,9] for a detailed discussion), and based on this identification we shall set up an "activation protocol" for indistinguishable particles.

The importance to study the correlations, particularly the entanglement, in terms of subalgebras of observables has been emphasized in [8,9], proving to be a useful approach for such analysis. The algebra of single-particle observables is generated by,

$$\mathcal{O}_{sp} = M \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \mathbb{I} \otimes M \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes M, \tag{6.26}$$

where M is an observable in the Hilbert space of a single particle. We can express this algebra in terms of fermionic or bosonic creation $\{a_i^\dagger\}_i$ and annihilation $\{a_i\}_i$ operators, depending on the nature of the particles in the system. The algebra is generated by quadratic observables $\mathcal{O}_{sp} = \sum_{ij} M_{ij} a_i^\dagger a_j$ that can be diagonalized as:

$$\mathcal{O}_{sp} = \sum_k \lambda_k \tilde{a}_k^\dagger \tilde{a}_k, \tag{6.27}$$

where $\tilde{a}_k^\dagger = \sum_j U_{kj} a_j^\dagger$ and U is the unitary matrix which diagonalizes M. Thus, since it is a non-degenerate algebra, the eigenvectors of their single-particle observables will be given by single Slater determinants, or permanents, for fermionic and bosonic particles respectively; more precisely, given by the set $\{\tilde{a}_{\vec{k}}^\dagger |vac\rangle\}_k$ where $\vec{k} = (k_1, \cdots, k_n)$, $k_i \in \{1, 2, ..., d\}$, represents the states of occupation of n particles:

$$\tilde{a}_{\vec{k}}^\dagger |vac\rangle = \tilde{a}_{k_1}^\dagger \tilde{a}_{k_2}^\dagger \cdots \tilde{a}_{k_n}^\dagger |vac\rangle, \tag{6.28}$$

where d is the single-particle dimension and $|vac\rangle$ is the vacuum state. The measurement of single-particle observables is therefore given by a von Neumann measurement, which we shall call hereafter as single-particle von Neumann measurement, according to the complete set of rank one projectors $\{\tilde{\Pi}_{\vec{k}} = \tilde{a}_{\vec{k}}^\dagger |vac\rangle \langle vac| \tilde{a}_{\vec{k}}\}_{\vec{k}}$, where they must satisfy:

$$\sum_{\vec{k}} \tilde{\Pi}_{\vec{k}} = \mathbb{I}_{\mathcal{A}(\mathcal{S})}. \tag{6.29}$$

being $\mathbb{I}_{\mathcal{A}} = \mathcal{A}$ the anti-symmetrization operator and $\mathbb{I}_{\mathcal{S}} = \mathcal{S}$ the symmetrization operator, they represent the identity on the fermionic and bosonic spaces. Let us consider the following notation, $\{a^{\dagger}_{\vec{k}} |vac\rangle\} = \left\{\left|f(\vec{k})\right\rangle\right\}$, $f(\vec{k}) \in \{1, 2, .., dim_{\mathcal{A}(\mathcal{S})}\}$, being f a bijective function of the sets $\{\vec{k}\}$ and $\{1, 2, .., dim_{\mathcal{A}(\mathcal{S})}\}$, and $dim_{\mathcal{A}(\mathcal{S})}$ is the dimension of the antisymmetric or symmetric subspaces.

To represent a projective measurement on the system, the interaction unitary U must act in order to perform the projective measurement Π with operators $\{\Pi_{\vec{k}} = a^{\dagger}_{\vec{k}} |vac\rangle \langle vac| a_{\vec{k}}\}$, where $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathcal{I}_{\mathcal{A}(\mathcal{S})}$.

Definition 136. *A unitary operator $U \in \mathcal{U}(\mathcal{F}^d_n)$ for fermionic systems, or $U \in \mathcal{U}(\mathcal{B}^d_n)$ for bosonic systems, it represents the interaction between the system of particles and the measurement apparatus during a local measurement, and its action is defined as:*

$$U\left|f(\vec{k})\right\rangle_{\mathcal{Q}} \otimes |j\rangle_{\mathcal{M}} = \left|f(\vec{k})\right\rangle_{\mathcal{Q}} \otimes \left|j \oplus f(\vec{k})\right\rangle_{\mathcal{M}}. \tag{6.30}$$

Given that the apparatus has at least the same dimension as the system.

It is easy to show that such operator is indeed unitary; note that

$$U = \sum_{\vec{k},j} \left|f(\vec{k})\right\rangle \left|j \oplus f(\vec{k})\right\rangle \left\langle f(\vec{k})\right| \langle j| , \tag{6.31}$$

thus,

$$UU^{\dagger} = \sum_{\vec{k},j,\vec{k}',j'} \delta_{\vec{k},\vec{k}'}\delta_{j,j'} \left|f(\vec{k})\right\rangle \left|j \oplus f(\vec{k})\right\rangle \left\langle f(\vec{k}')\right| \left\langle j' \oplus f(\vec{k}')\right| , \tag{6.32}$$

and since $\{\left|f(\vec{k})\right\rangle_{\mathcal{Q}}\}_{\vec{k}}$ and $\{\left|j \oplus f(\vec{k})\right\rangle_{\mathcal{M}}\}_j$ form a complete set, we have that $UU^{\dagger} = \mathbb{I}_{\mathcal{A}(\mathcal{S})} \otimes \mathbb{I}_{\mathcal{M}}$.

Now we can use the approach of the activation protocol to find the set of classical correlated state in the system of indistinguishable particles.

Theorem 137 (States without quantumness of correlations [86])**.** *Considering a n-partite system indistinguishable particles, represented by the state $\xi \in \mathcal{D}(\mathcal{F}^d_n)$ for fermions and $\xi \in \mathcal{D}(\mathcal{B}^d_n)$ for bosons. This state has no quantumness of correlations if and only if the exist an local orthonormal basis as:*

$$\xi = \sum_{\vec{k}} p_{\vec{k}}\, \tilde{a}^{\dagger}_{\vec{k}} |vac\rangle \langle vac| \tilde{a}_{\vec{k}}, \quad \sum_{\vec{k}} p_{\vec{k}} = 1, \tag{6.33}$$

where $\tilde{a}^{\dagger}_{\vec{k}} |vac\rangle = V^{\otimes n} a^{\dagger}_{\vec{k}} |vac\rangle$, V is a unitary matrix, and $\{a^{\dagger}_{\vec{k}}\}$ an orthonormal set of creation operators

Proof. We shall first show that states given by Eq.(6.33) do not generate entanglement, and then that they are the only ones. Let U be the coupling unitary corresponding to the measurement Π with measurement operators $\{\Pi_{\vec{k}} = a^{\dagger}_{\vec{k}} |vac\rangle \langle vac| a_{\vec{k}} = \left|f(\vec{k})\right\rangle \left\langle f(\vec{k})\right|\}_{\vec{k}}$, where $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathbb{I}_{\mathcal{A}(\mathcal{S})}$. Applying the activation protocol on states given by Eq.(6.33), using $\bar{V} = V^{\dagger}$ as the single-particle unitary transformation, it follows that:

$$\begin{aligned} \rho_{\mathcal{Q}:\mathcal{M}} &= U\left[(\bar{V}^{\otimes n}\xi\bar{V}^{\dagger^{\otimes n}})_{\mathcal{Q}} \otimes |0\rangle\langle 0|_{\mathcal{M}}\right] U^{\dagger} \\ &= \sum_{\vec{k}} p_{\vec{k}} \left|f(\vec{k})\right\rangle\left\langle f(\vec{k})\right|_{\mathcal{Q}} \otimes \left|f(\vec{k})\right\rangle\left\langle f(\vec{k})\right|_{\mathcal{M}}, \end{aligned} \tag{6.34}$$

where $\rho_{\mathcal{Q}:\mathcal{M}} \in Sep(\mathcal{Q} \otimes \mathcal{M})$. The demonstration that such states correspond to the unique states that do not generate entanglement is given below. A separable state between system and measurement apparatus is given as:

$$\sigma = \sum_{i} p_i \, |\psi_i\rangle\langle\psi_i|_{\mathcal{Q}} \otimes |\phi_i\rangle\langle\phi_i|_{\mathcal{M}} , \tag{6.35}$$

noting that the sets $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ are not necessarily orthogonal. Since the activation protocol corresponds to a unitary operation, thus invertible, there must exist a set $\{|\eta_i\rangle\}$ of states for the system such that,

$$U(V^{\otimes n})\ |\eta_i\rangle_{\mathcal{Q}} \otimes |0\rangle_{\mathcal{M}} = |\psi_i\rangle_{\mathcal{Q}} \otimes |\phi_i\rangle_{\mathcal{M}}\,, \tag{6.36}$$

and $\rho_{\mathcal{Q}} = \sum_i p_i\ |\eta_i\rangle\langle\eta_i|$. Expanding $\{|\eta_i\rangle\}$ on the basis $\{a^\dagger_{\vec{k}}\,|vac\rangle\}$ "transformed" by $V^{\dagger\otimes n}$,

$$|\eta_i\rangle = \sum_{\vec{k}} c^{(i)}_{\vec{k}}\ V^{\dagger\otimes n}\ a^\dagger_{\vec{k}}\,|vac\rangle\,, \tag{6.37}$$

we see from Eqs.(6.36) and (6.37) that,

$$U(V^{\otimes n})\ |\eta_i\rangle \otimes |0\rangle = \sum_{\vec{k}} c^{(i)}_{\vec{k}}\ a^\dagger_{\vec{k}}\,|vac\rangle \otimes \left|f(\vec{k})\right\rangle = |\psi_i\rangle \otimes |\phi_i\rangle\,. \tag{6.38}$$

The above factorization condition imposes the following restriction: $c^{(i)}_{\vec{k}} = \gamma_i\,\delta_{\{\vec{k},g(i)\}}$, $\|\gamma_i\| = 1$, $g : \{i\} \mapsto \{\vec{k}\}$. Therefore,

$$\begin{aligned}
\rho_{\mathcal{Q}} &= \sum_i p_i\ |\eta_i\rangle\langle\eta_i|\,,\\
&= \sum_i p_i\,(\sum_{\vec{k}} \gamma_i\,\delta_{\{\vec{k},g(i)\}}\ a^\dagger_{\vec{k}}\,|vac\rangle)\\
&\quad (\sum_{\vec{k}'} \langle vac|\,a_{\vec{k}'}\,\gamma_i^*\,\delta_{\{\vec{k}',g(i)\}}),\\
&= \sum_i p_i\ \underbrace{\|\gamma_i\|}_{1}\ a^\dagger_{g(i)}\,|vac\rangle\langle vac|\,a_{g(i)},
\end{aligned} \tag{6.39}$$

i.e, the states with no quantumness of correlations as given by Eq.6.33. □

Example. Let us show an example of the approach in order to clarify the formalism and the above analysis. An interesting case concerns to the controversial bosonic quantum state $|\psi_b\rangle \in \mathcal{B}^2_2$, written as:

$$|\psi_b\rangle = \frac{1}{2}(b^\dagger_0 b^\dagger_0 + b^\dagger_1 b^\dagger_1)\,|vac\rangle\,, \tag{6.40}$$

where $\{b^\dagger_i\}$ are the bosonic creation operators. Such a state is considered both entangled by some authors [56, 69, 70, 119], as non entangled for others [9, 64, 65, 97]. Note that such a state can actually be described by a single Slater permanent $|\psi_b\rangle = b^\dagger_+ b^\dagger_-\,|vac\rangle$, being $b^\dagger_\pm = \frac{1}{\sqrt{2}}(b^\dagger_0 \pm b^\dagger_1)$. Defining the coupling unitary U corresponding to the $\{\Pi_{\vec{k}} = b^\dagger_{\vec{k}}\,|vac\rangle\langle vac|\,b_{\vec{k}}\}$, $\sum_{\vec{k}} \Pi_{\vec{k}} = \mathbb{I}_{\mathcal{S}}$, $\{\vec{k}\} = \{(0,0),(0,1),(1,1)\}$, and using the notation,

$$b^\dagger_0 b^\dagger_0\,|vac\rangle = |0\rangle\,,\ b^\dagger_0 b^\dagger_1\,|vac\rangle = |1\rangle\,,\ b^\dagger_1 b^\dagger_1\,|vac\rangle = |2\rangle\,, \tag{6.41}$$

we have that the unitary acts as follows,

$$U\,|k\rangle_{\mathcal{Q}} \otimes |0\rangle_{\mathcal{M}} = |k\rangle_{\mathcal{Q}} \otimes |k\rangle_{\mathcal{M}}\,. \tag{6.42}$$

Applying this unitary on the bosonic state, we generate an entangled state between system and apparatus, $U(|\psi_b\rangle_{\mathcal{Q}} \otimes |0\rangle_{\mathcal{M}}) = \frac{1}{2}(b^\dagger_0 b^\dagger_0\,|vac\rangle \otimes |0\rangle + b^\dagger_1 b^\dagger_1\,|vac\rangle \otimes |2\rangle)$, but this is not a unavoidable entanglement in order to realize that measurement, since we could apply, before the unitary coupling, the following single-particle unitary transformation: $V : |+\rangle = |0\rangle + i\,|1\rangle \mapsto |0\rangle$, and $V : |-\rangle = |0\rangle - i\,|1\rangle \mapsto |1\rangle$, then applying on the Fock space of the bosons:

$$V \otimes V : \begin{cases} b^\dagger_+ \mapsto b^\dagger_0,\\ b^\dagger_- \mapsto b^\dagger_1. \end{cases} \tag{6.43}$$

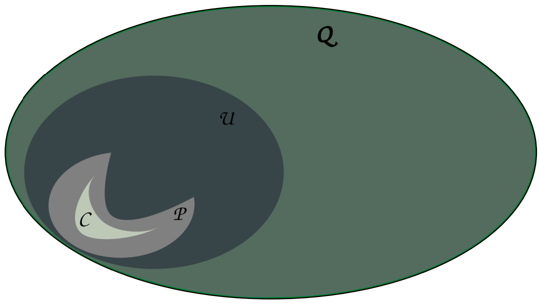

Figure 6.1: *(Color online) Schematic picture of the distinct types of correlations in systems of indistinguishable particles. The largest set ($\mathcal{Q}$) denotes the set of all fermionic, or bosonic, quantum states; the blue area ($\mathcal{U}$) represents the convex set of states with no entanglement; the gray area ($\mathcal{P}$) represents the non convex set of states with no quantumness of correlations, as defined in this article (Eq.(6.33)); and the yellow area ($\mathcal{C}$) represents the non convex set of states with no exchange correlations due to the particle statistics, possessing only classical correlations. Note that for fermionic particles, the set $\mathcal{C}$ is a null set. The following hierarchy is identified: $\mathcal{C} \subset \mathcal{P} \subset \mathcal{U} \subset \mathcal{Q}$.*

We see now that the coupling between system and apparatus does not generate entanglement between them, $U\left[(V \otimes V)\left|\psi_b\right\rangle_{\mathcal{Q}} \otimes |0\rangle_{\mathcal{M}}\right] = U(b_0^\dagger b_1^\dagger |vac\rangle_{\mathcal{Q}} \otimes |0\rangle_{\mathcal{M}}) = b_0^\dagger b_1^\dagger |vac\rangle_{\mathcal{Q}} \otimes |1\rangle_{\mathcal{M}} \in Sep(\mathcal{Q} \otimes \mathcal{M})$, and thus such a state has no quantumness of correlations.

The correlations between indistinguishable particles can thereby be characterized by different types: the entanglement, the quantumness of correlations, the correlations generated merely by particle statistics (exchange correlation), and the classical correlations. In fact, there are quantum states whose particles are classically correlated, not even possessing exchange correlations, such as pure bosonic states with all their particles occupying the same degree of freedom, $|\psi_b\rangle = \frac{1}{\sqrt{n!}}(b_i^\dagger)^n |vac\rangle$, or mixed states described by an orthonormal convex decomposition of such pure states, $\chi_b = \sum_i \frac{1}{n}(b_i^\dagger)^n |vac\rangle \langle vac| (b_i)^n$. See Fig.6.1 for a schematic picture of these different kinds of correlations. Interesting questions to raise concern how the notion of entanglement of particles is related to the quantumness of correlations, and if they are equivalent for pure states. We can note from Eq.6.33 that, for pure states, the set with no quantumness of correlations is described by states with a single Slater determinant, or permanent, which is equivalent to the set of unentangled pure states. Actually there is an ongoing debate regarding the correct definition of particle entanglement [8,9,56,64,65,97,139,140], but at the same time there are strong physical reasons to consider particle entanglement in pure states as the correlations beyond the mere exchange correlations [8,9,65,97,139,140]. Concerning mixed states, it becomes clear that the set given by Eq.6.33 is a subset of the unentangled one, thereby being quantumness of correlations a more general notion of correlations than entanglement.

Quantifying quantumness of correlations for indistinguishable particles

According to the activation protocol, different entanglement measures will lead, in principle, to different quantifiers for the quantumness of correlations. We can thus define the measure Q_E for quantumness of correlations, associated with the entanglement measure E, as follows [86].

Definition 138. *Considering a system of n particles sharing a d-dimensional single particle space, described by the state $\rho_{\mathcal{Q}}$. We can define a given measure of quantumness of correlations Q_E as:*

$$Q_E(\rho_{\mathcal{Q}}) = \min_V E(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}), \tag{6.44}$$

where $\tilde{\rho}_{\mathcal{Q},\mathcal{M}} = U\left[(V^{\otimes n}\rho_{\mathcal{Q}} V^{\dagger^{\otimes n}}) \otimes |0\rangle\langle 0|_{\mathcal{M}}\right] U^\dagger$.

We shall consider two different entanglement measures for the bipartite entanglement, the physically motivated distillable entanglement E_D [19] and the relative entropy of entanglement

E_r [155, 156]. Note that the output states of the activation protocol have the so called maximally correlated form [134] between system and measurement apparatus.

Lemma 139. *The state after the interaction between the system and the measurement apparatus is a maximally correlated state:*

$$\tilde{\rho}_{\mathcal{Q},\mathcal{M}} = \sum_{\vec{l},\vec{l}'} \chi^V_{\vec{l},\vec{l}'} \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|_{\mathcal{Q}} \otimes \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|_{\mathcal{M}}, \tag{6.45}$$

being $\chi^V_{\vec{l},\vec{l}'} = (\Pi^V_{\vec{l}})^\dagger \rho_{\mathcal{Q}} (\Pi^V_{\vec{l}'})$, *where* $\Pi^V_{\vec{l}} = V^{\otimes n}\Pi_{\vec{l}}$.

Proof. Let us show that the output states of the activation protocol for indistinguishable particles have the so called maximally correlated form between system and measurement apparatus. If $\{a^\dagger_{\vec{k}}\,|vac\rangle\} = \left\{\left|f(\vec{k})\right\rangle\right\}$ is the system basis, U is the coupling unitary given by Eq.(6.30), and V is the unitary respective to the single particle transformation, we have that,

$$\begin{aligned} V^{\otimes n}\, a^\dagger_{\vec{k}}\,|vac\rangle &= (\sum_{l_1} v_{k_1 l_1} a^\dagger_{l_1}) \cdots (\sum_{l_n} v_{k_n l_n} a^\dagger_{l_n})\,|vac\rangle\,, \\ &= \sum_{\vec{l}} v_{k_1 l_1} \cdots v_{k_n l_n} \left|f(\vec{l})\right\rangle, \end{aligned} \tag{6.46}$$

where $v_{k_i l_j}$ are the matrix elements of V. A general state for the system is given by:

$$\rho_{\mathcal{Q}} = \sum_{\vec{k},\vec{k}'} p_{\vec{k},\vec{k}'} \left|f(\vec{k})\right\rangle\left\langle f(\vec{k}')\right|; \tag{6.47}$$

thereby,

$$\begin{aligned} V^{\otimes n}\rho_{\mathcal{Q}}V^{\dagger\otimes n} &= \sum_{\vec{k},\vec{k}',\vec{l},\vec{l}'} p_{\vec{k},\vec{k}'} (v_{k_1 l_1}\cdots v_{k_n l_n})\,(v_{k'_1 l'_1}\cdots v_{k'_n l'_n})^\dagger \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|, \\ &= \sum_{\vec{l},\vec{l}'} \chi^V_{\vec{l},\vec{l}'} \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|, \end{aligned} \tag{6.48}$$

where $\chi^V_{\vec{l},\vec{l}'} = \sum_{\vec{k},\vec{k}'} p_{\vec{k},\vec{k}'} (v_{k_1 l_1}\cdots v_{k_n l_n})\,(v_{\vec{k}'_1 l'_1}\cdots v_{\vec{k}'_n l'_n})^\dagger$. The output states of the activation protocol thus have the form

$$\begin{aligned} \rho_{\mathcal{Q}:\mathcal{M}} &= U\left[(V^{\otimes n}\rho_{\mathcal{Q}}V^{\dagger^{\otimes n}})\otimes|0\rangle\langle 0|_{\mathcal{M}}\right]U^\dagger \\ &= \sum_{\vec{l},\vec{l}'} \chi^V_{\vec{l},\vec{l}'} \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|_{\mathcal{Q}} \otimes \left|f(\vec{l})\right\rangle\left\langle f(\vec{l}')\right|_{\mathcal{M}}, \end{aligned} \tag{6.49}$$

i.e., the maximally correlated form. □

If the entanglement between the measurement apparatus and system during the local measurement process is quantified via the distillable entanglement, the related measure of quantumness of correlations is the relative entropy of quantumness for indistinguishable particles.

Proposition 140. *Given a system of indistinguishable particles described by the state* $\rho_{\mathcal{Q}}$, *distillable entanglement between the system and the apparatus results in the relative entropy of quantumness:*

$$Q_{E_{D(r)}}(\rho_{\mathcal{Q}}) = \min_{\chi\in\Omega} S(\rho_{\mathcal{Q}} \parallel \chi), \tag{6.50}$$

where $S(\rho \parallel \chi) = Tr(\rho\ln\rho - \rho\ln\chi)$ *is the relative entropy and* Ω *is the set of states without quantumness of correlations.*

Proof. The entanglement for maximally correlated states according to the distillable entanglement [78], as well as for the relative entropy of entanglement [134], is given by:

$$E_{D(r)}(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}) = S(\tilde{\rho}_{\mathcal{Q}}) - S(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}), \tag{6.51}$$

where $S(\rho) = -Tr(\rho \ln \rho)$ is the von Neumann entropy. The first term is given by:

$$S(\tilde{\rho}_{\mathcal{Q}}) = S\left(\sum_{\vec{l}} (\Pi^V_{\vec{l}})^\dagger \rho_{\mathcal{Q}} (\Pi^V_{\vec{l}}) \left| f(\vec{l}) \right\rangle \left\langle f(\vec{l}) \right| \right), \tag{6.52}$$

i.e.,the entropy of the projected state $\rho_{\mathcal{Q}}$ according to a single-particle von Neumann measurement. The second term is simply:

$$S(\tilde{\rho}_{\mathcal{Q},\mathcal{M}}) = S(U[V^{\otimes n} \rho_{\mathcal{Q}} \otimes |0\rangle\langle 0|_{\mathcal{M}} V^{\dagger\otimes n}]U^\dagger) = S(\rho_{\mathcal{Q}}), \tag{6.53}$$

since it is invariant under unitary transformations. Thus we have that the quantumness of correlations measure is given by,

$$Q_{E_{D(r)}}(\rho_{\mathcal{Q}}) = \min_V \left[S\left(\sum_{\vec{l}} (\Pi^V_{\vec{l}})^\dagger \rho_{\mathcal{Q}} (\Pi^V_{\vec{l}}) \left| f(\vec{l}) \right\rangle \left\langle f(\vec{l}) \right| \right) - S(\rho_{\mathcal{Q}}) \right], \tag{6.54}$$

which corresponds to the notion of minimum disturbance caused in the system by single-particle measurements. This result is in agreement with the analysis made in [107] for the particular case of two-fermion systems, and to the best of our knowledge is the only study attempting to characterize and quantify a more general notion of correlations between indistinguishable particles. Repeating the calculations in Theorem 99, it is possible to obtain that the Eq.(6.54) is an equivalent expression to:

$$Q_{E_{D(r)}}(\rho_{\mathcal{Q}}) = \min_{\chi\Omega} S(\rho_{\mathcal{Q}} \parallel \chi), \tag{6.55}$$

□

The above equation introduces a geometrical approach to the particle correlation measure. Notably we see that, as well as for the quantumness of correlations in distinguishable subsystems, the correlations between indistinguishable particles defined in this section has a variety of equivalent approaches in order to characterize and quantify it, as shown by the activation protocol (Eq.6.44), minimum disturbance (Eq.6.54) and geometrical approach(Eq.6.55).

The set of states without quantumness of correlations for indistinguishable particles in Eq.6.33 can be obtained applying the symmetrization or anti-symmetrization operation on the set of strictly classical correlated states for distinguishable particles. As a simple symmetrization or anti-symmetrization cannot create or destroy quantum correlations, it creates a bijective map between the set of strictly classical correlated states in the space of states and the set of classical correlated states in the Fock space. Although the states in Eq.6.33 do not have any kind of quantum correlations, they cannot be treated like classical probability distributions, since they respect quantum rules: like the Pauli exclusion principle. Given the geometric approach, and the set of states without quantumness of correlations obtained from the activation protocol we can define an analogous to the geometric discord for indistinguishable particles. This measure of quantumness of correlations for indistinguishable particles was proposed for fermions, before we had obtained the result in Theorem 137, where we define the set of states without quantumness of correlations as the antisymmetrization of the distinguishable set of classical correlated states [87].

Definition 141. *Consider a system composed by n particles with d degrees of freedom each, described by the state $\rho \in \mathcal{D}(\mathcal{F}^d_n)$ for fermions, and $\rho \in \mathcal{D}(\mathcal{B}^d_n)$ for bosons. The geometric measure of quantum discord can be written as [87]:*

$$\mathcal{D}_1(\rho) = \min_{\xi\in\Omega} \|\xi - \rho\|_1, \tag{6.56}$$

where Ω *is the set of states without quantumness of correlations for indistinguishable particles.*

The geometrical discord for identical particles has the same properties of the the geometrical discord via 1-norm discussed in Chapter 3. As for indistinguishable particles we cannot apply the measurement in just one system, without affect the others, because the particles cannot be distinguished, the monotonic property of the trace distance is not valid.

Chapter 7

Conclusions and future perspectives

The quantumness of correlations revealed very interesting for quantum information area, not only by the role in some quantum informational and quantum computational tasks, but also because its fundamental meaning in quantum mechanics. It is related to some fundamental aspects of quantum mechanics, as well the mathematical structure of it. Quantumness of correlations is related to the superposition principle of quantum mechanics, the measurement process and the composition of quantum systems. Actually it comes from the incapacity to extract all the information shared by two parts of a same system. This incapacity is related to the way that the quantum measurement is performed on the quantum systems, as a classical statistical inference process. In the thesis we always approached quantum measurements as quantum to classical channels, because in this way it is clear how fundamental is the quantumness of correlations, and this foundations are related to destruction of the quantum character of the system during the measurement process. We can illustrate what this means comparing two distinct situations. Considering, as example, a source which prepares always the same quantum state, if we would like to know what state is created we can perform any tomographic method to recovery the state. However, considering now the source is creating two states according a classical random variable. If the states are not orthogonal, does not exist a measurement process capable to distinguish them. This comes from the fact that there exist quantum correlations between the classical random variable and the quantum source. These quantum correlations are completely destroyed by the measurement process, it also destroys the quantum characteristics of the system. Calculating a measure of quantumness of correlations, quantum discord for example, we are obtaining the degree of quantumness of the systems, in the correlations point of view, in other words, quantum discord is given to us the amount of quantumness which will be destroyed by the measurement process.

In this thesis we approached quantumness of correlation in three different ways. Indeed we walked on some fundamental descriptions for quantumness of correlations: geometrical approach, and its relation with quantum entanglement; entropic approach, and its role in the discrimination of quantum states and accessibility of information; and the indistinguishable particle approach, where we calculate the class of states without quantumness of correlation for identical particles from the entanglement between distinguishable systems. This thesis is a journey in the quantum world in its most foundational form.

Our original contributions started to appear in the thesis in Chap.3, where we introduced a general geometrical measure for quantumness of correlations via Schatten-p norm. The 1-norm geometrical quantum discord deserves a special attention, once that after our proposal people proved that it is the only geometrical measure which satisfies the entropic properties of quantum discord. In Chap.4 we revise two different ways to compare quantumness of correlations and quantum entanglement. The Koash-Winter relation associates quantum discord in a bipartite system with the entanglement created, between the system and purification environment, in the purification process. We also present the activation protocol, which relates quantumness of correlations with the entanglement created between the system and the apparatus during the measurement process. Then we present an original relation between a general geometrical measure of quantumness of correlations and the quantum entanglement, calculated via entanglement witness. We adopt a kind of rule to compare these two quantities, as both are calculated via a geometrical approach. In Chap.5 we compared the capacity to access the information of a G-covariant ensemble via projective measurements and POVM, we obtained an expression for qubit systems, where we state that even for this simple case POVM are needed. It reflects on quantum discord for classical-quantum states, that is equal to the amount of information which cannot be extracted from the ensemble. The result obtained in this section is counterintuitive, because we do not expect that the measurement performed to optimize quantum discord depends on the dimension of the unmeasured system. In the context of indistinguishable particles we defined a projective measurement on the ensemble of states as a dephasing on the optimal POVM. Given this definition we calculated that 1-norm geometrical quantum discord, of the composed by a classical random variable and the states of the ensemble, is a lower bound for the difference in the probability o success, to distinguish the states, via POVM or the dehased-POVM. We introduced a measure of projectiveness of a given POVM, we obtained a bound between this value and the difference in the probability of success for POVM or the dephased-POVM. We investigate these concepts for a given POVM named pretty good measurement, and calculated the difference in the probability of success via PGM and the dephased-PGM for a 2-design ensemble. In the end of this chapter we discuss an open question about the 1-way work deficit under embedding. The last chapter is the Chap.6. In this chapter we presented the formalism of identical particles. We introduced an approach for the separable states in this context and proposed a measure of entanglement for fermions. The measure of entanglement calculated by us is the fermionic version of the generalized robustness of entanglement. Indeed it is not well understood in the literature who are the separable states for indistinguishable particles, there exist more than one approach and interpretation for it. Our description is supported by the following statement: quantum correlations cannot be created just symmetrizing the space of states. This kind of argument supports the approach discussed in this chapter. In this chapter we also discussed the concept of quantumness of correlation for indistinguishable particles. We calculated who are the states without quantumness of correlation, in this context, via the activation protocol. Then we obtained that the states which do not create entanglement with the measurement apparatus during a local measurement are the states without quantumness of correlations. These states has only exchange correlations, which are the correlations created by symmetrization of the Hilbert space. Therefore, as entanglement is a subclass of quantumness of correlations, we know what are the states that are not entangled, if they are states without quantumness of correlations. We discussed a bosonic state, which is approached in the literature as an entangled state, although it is a state without quantumness of correlations. We performed the same formalism to obtain the relative entropy of quantumness for identical particles. We believe that this work will help people to understand what is the meaning of quantum correlations for identical particles systems. These are the contributions performed by us in this phd thesis.

As start to work with science means create more work, in this thesis it is not different. We have some open problems and subjects that can be interesting to investigate.

- In Chap.5 we introduced the dephased-PGM, then it can be interesting to obtain an interplay between the probability of success for the PGM and its dephased version, which depends exclusively in the ensemble. Given this expression we can use for example the Fano's inequality [162] and the definition of max-entropy [90] to obtain an upper bound for the difference in the quantum discord calculated via POVM and projective measurements for classical-quantum states, and a function of the probability of success to distinguish the states via PGM;

- continuing in Chap.5 we have a conjecture that the diamond norm of the measurement map via POVM and the dephasing channel, which optimize the quantum discord, is very smaller than 1. Proving the conjecture we can use the Alicki-Fannes inequality [162] to obtain a general upper bound for the difference in the quantum discord via POVM and projective measurement;

- an open problem which interesting me is calculate the answer for the question: i the 1-way work deficit embedding free?

- it is known that Koash-Winter (KW) relation reveals many important properties about the monogamy of quantum correlations. However quantumness of correlations are related to local measurements in composed system, then it is not clear, in the literature, a foundational interpretation for the KW relation. I would like to study a possible generalization for other measures than entanglement of formation, may be a way to obtain it is via the generalized entanglement of formation, which contain as a subclass the negativity and the concurrence [14]. Another approach for the KW that I am interested is via the activation protocol [129];

- finally, it is known that the quantumness of correlation can increase under the action of local maps applied on the measured system. Then the following question arises from this fact: what are the operation such the quantumness of correlations are monotonic? Two interesting classes to study are the unital operations and the mixed unitary operations. It is know that for two qubits systems the quantum discord is monotonic under the action of the dephasing channel [151], also for qubits any unital map is mixed unitary, although it is not true in general [159].

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