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A MEASURE-THEORETIC DESCRIPTION OF THE SPECTRAL TYPES OF THE ALMOST MATHIEU OPERATOR

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A measure-theoretic description of the spectral types of the almost Mathieu operator

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Abstract

The almost Mathieu operator is one of the most studied quasiperiodic Schrödinger operators and arises as a physical model for the electron in certain systems. At first, it was conjectured [1] that the spectral type of this operator with irrational frequencies would depend only on the coupling constant, being pure point when it is greater than 1 and purely absolutely continuous when it is less than 1. Although wrong as stated, this claim does hold in a measure-theoretic sense. In this work, we explore the studies of the almost Mathieu operator carried so far to show that such classification of the spectrum holds for full measure sets of frequencies and phases.

Keywords: almost Mathieu operator; quasiperiodic Schrödinger operators; spectral theory.

Resumo

O operador almost Mathieu é um dos operadores de Schrödinger quasiperiódicos mais estudados e surge como um modelo físico para o elétron em determinados sistemas. A princípio, conjecturou-se [1] que o tipo espectral de tal operador com frequências irracionais dependesse apenas da constante de acoplamento: o espectro seria puramente pontual quando ela fosse maior que 1 e puramente absolutamente contínuo quando ela fosse menor que 1. Apesar de essa conjectura ser falsa dessa forma, ela é válida em um ponto de vista de Teoria da Medida. Neste trabalho, nós usamos os estudos sobre o operador almost Mathieu feitos até então para mostrar que tal classificação do espectro é verdadeira em conjuntos de medida total de frequências e fases.

Palavras-chave: operador almost Mathieu; operadores de Schrödinger quasiperiódicos; teoria espectral.

Contents

1 Introduction

Throughout this text, ℓ^2 will denote the Hilbert space $\left\{ u : \mathbb{Z} \to \mathbb{C} \;\middle|\; \sum_{n\in\mathbb{Z}} |u(n)|^2 < \infty \right\}$. Given real numbers λ, α, θ, we define the **Almost Mathieu Operator** $H_{\lambda,\alpha,\theta} : \ell^2 \to \ell^2$ as the bounded self-adjoint linear operator given by

$$[H_{\lambda,\alpha,\theta}u](n) = u(n+1) + u(n-1) + 2\lambda\cos\left[2\pi(n\alpha+\theta)\right]u(n). \tag{1.1}$$

The operator $H_{\lambda,\alpha,\theta}$ is a tight biding model for a electron in an one dimensional lattice and is also related to the quantum Hall effect [21, 23]. The parameters λ, α, θ are known as coupling constant, frequency and phase, respectively. Since $\cos(x+2\pi n) = \cos(x)$ and $-\cos(x) = \cos(x+\pi)$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, there is no loss of generality in considering $\alpha, \theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\lambda \geq 0$. Furthermore, when $\alpha \in \mathbb{Q}$ or $\lambda = 0$, the operator has a periodic potential, so the properties of its spectrum are well known from Floquet theory. Therefore, we will be interested only in irrational frequencies and positive coupling. We say that the coupling is **subcritical** if $0 < \lambda < 1$; **critical** if $\lambda = 1$; and **supercritical** if $\lambda > 1$. For $x \in \mathbb{R}$, we will denote by $\|x\|_{\mathbb{T}}$ the distance of x to $\mathbb{Z}$.

One advantage of dealing with irrational frequencies is that the spectrum is phase-independent:

Theorem 1.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $\theta \in \mathbb{R}$. Let $H_{f,\alpha,\theta}$ be defined by*

$$[H_{f,\alpha,\theta}u](n) = u(n+1) + u(n-1) + f(n\alpha+\theta)u(n).$$

Then for every $\theta \in \mathbb{R}$, the equality $\sigma(H_{f,\alpha,\theta}) = \sigma(H_{f,\alpha,0})$ holds.

Proof. Given $k \in \mathbb{Z}$, define $U_k : \ell^2 \to \ell^2$ by $[U_k u](n) = u(n+k)$. Then U_k is unitary with $U_k^* = U_{-k}$ and $U_k H_{f,\alpha,0} U_k^* = H_{f,\alpha,k\alpha}$, so $\sigma(H_{f,\alpha,k\alpha}) = \sigma(H_{f,\alpha,0})$ for every $k \in \mathbb{Z}$.

Now, since α is irrational, $\{k\alpha \mod 1\}_{k\in\mathbb{Z}}$ is dense in $\mathbb{T}$. Given $\theta \in \mathbb{R}$, $\varepsilon > 0$, pick $\delta > 0$ such that $\|\theta_1 - \theta_2\|_{\mathbb{T}} < \delta \implies |f(\theta_1) - f(\theta_2)| < \varepsilon$ and then choose $m \in \mathbb{Z}$ such that $\|\theta - m\alpha\|_{\mathbb{T}} < \delta$. We then get

$$\left|\left[(H_{f,\alpha,m\alpha} - H_{f,\alpha,\theta})u\right](n)\right| = \left|f(n\alpha+m\alpha) - f(n\alpha+\theta)\right| |u(n)| < \varepsilon|u(n)|.$$

So $\|H_{f,\alpha,m\alpha} - H_{f,\alpha,\theta}\| \leq \varepsilon$. The theorem is then immediate from the folowing result of Perturbation Theory (see [27]):

Proposition 1.2. *If H_1 and H_2 are bounded self-adjoint operators and d_{haus} denote the Hausdorff distance, then*

$$d_{haus}(\sigma(H_1), \sigma(H_2)) \leq \|H_1 - H_2\|.$$

□

Although the set $\sigma(H_{\lambda,\alpha,\theta})$ does not depend on θ, the absolutely continuous, singular continuous and point components in general do depend on it. Throughout this text we will give the classification of the spectrum of the almost Mathieu operator based on methods that (as the title sugests) exclude sets of frequencies and of phases of measure zero. We make the following path:

In chapter 2, using [20] we explore a powerful duality relation between $H_{\lambda,\alpha,\theta}$ and $H_{\lambda^{-1},\alpha,\theta}$ and show how their spectral components are related.

In chapter 3, using [24] we show that for almost every frequency and almost every phase the supercritical almost Mathieu operator has only point spectrum and, as a consequence of the results of chapter 2, we get that the subcritical almost Mathieu operator has only absolutely continuous spectrum for almost every frequency and almost every phase. Using [8, 26, 32] we also show that in the supercritical case the result cannot be extended to *every* frequency or *every* phase.

In the first section of chapter 4, we use the work of Last [30] to show that the measure of the spectrum of the almost Mathieu operator depends only on the coupling constant (at least for α in a full measure dense G_δ set of real numbers), and in particular that $\sigma(H_{1,\alpha,\theta})$ has zero measure for such α's. In the next section we then use the work of Avila [3] to show that $H_{1,\alpha,\theta}$ cannot have eigenvalues expect maybe for countably many phases, which yields that $\sigma(H_{1,\alpha,\theta})$ is purely singular continuous for almost every α and all but possibly countably many θ.

In chapter 5, for completeness, we present some improved results that could not be shown in details in this work.

At first sight, the study of the almost Mathieu operator seems very specific. Nevertheless, as Theorem 1.1 suggests, many of the results we will study in this text can be applied to a broader class of quasiperiodic Schrödinger operators. The analyticity of the cossine function and the self-duality that will be treated in chapter 2 do allow us to make some simpifications for the case of the almost Mathieu operator, but one of the objectives of the studies that led to the writing of this material was precisely to gather techniques to study open problems involving the spectral theory of more general quasiperiodic Schrödinger operators.

The appendices are dedicated to some basic definitions and results of different areas of Mathematics that are relevant to the text. In Appendix A, we state some results

of Spectral Theory for bounded self-adjoint operators; in Appendix B, we give the basic definitions of Ergodic Theory and state two ergodic theorems and the Denjoy-Koksma inequality; in appendix C, we discuss about the transfer matrices for eigenvalue equations of Schrödinger operators on ℓ^2; and in appendix D, we talk about the approximation of real numbers by rational continued fractions and some properties of diophantine numbers.

Throughout this text ν will denote the Lebesgue measure and expressions such as "almost everywhere" and "measure zero", when the measure is not explicitly stated, will refer to Lebesgue measure.

2 Duality

This chapter aims to study the relation between the spectral properties of $H_{\lambda,\alpha,\theta}$ and those of $H_{\lambda^{-1},\alpha,\theta}$. It is entirely based on the work of Gordon, Jitomirskaya, Last and Simon in [20].

In the first section, we show that for fixed λ and α we can treat the set of eigenvalues and eigenvectors of $H_{\lambda,\alpha,\theta}$ as a measurable function of θ (with respect to the Borel $\sigma-$algebra in $\mathbb{T}$).

In the second section, we explore a version of Aubry-André duality, where we prove that the direct integral of $H_{\lambda,\alpha,\theta}$ in θ is unitarily equivalent to the direct integral of $\lambda H_{\lambda^{-1},\alpha,\theta}$ in θ. We also show that the spectral measure of $H_{\lambda,\alpha,\theta}$ with respect to a fixed vector is almost everywhere independent of θ.

In the third section, using a result of Deift and Simon [17], we show that for almost every $\theta, \theta' \in \mathbb{T}$ the singular component of a spectral measure for $H_{\lambda,\alpha,\theta}$ and the singular component of a spectral measure for $H_{\lambda,\alpha,\theta'}$ are mutually singular.

In the fourth section, at last, we use the results shown previously to show how point spectrum at $H_{\lambda,\alpha,\theta}$ yields absolutely continuous spectrum at $H_{\lambda^{-1},\alpha,\theta}$.

2.1 Measurability of eigenvalues and eigenfunctions

Consider that an irrational α and a positive λ are fixed, and given $\theta \in \mathbb{T}$ denote $H_{\lambda,\alpha,\theta}$ simply by H_θ. Given an eigenfunction u of H_θ (if there is any) such that $\|u\| = 1$, define $j(u)$ as the leftmost maximum of u, that is, j is the only integer such that $|u(j)| \geq |u(k)| \, \forall k \in \mathbb{Z}$ and $|u(j)| > |u(k)| \, \forall k < j$. The existence of such j is clear since $u \in \ell^2$ implies that $|u(n)|$ has a maximum value and that it can be the image of only finitely many points. If j is the leftmost maximum of u, we say that u is **attached** to j. From now on, given an eigenfunction u attached to j, we will fix u such that $u(j) > 0$.

Let $N_j(\theta)$ be the number of eigenvectors of H_θ attached to $j \in \mathbb{Z}$. Define for each $k \in \mathbb{N}$ the set $\mathbb{T}_{j,k} = \{\theta \in \mathbb{T} \,|\, N_j(\theta) \geq k\}$. Clearly $\mathbb{T}_{j,k+1} \subset \mathbb{T}_{j,k}$. Using Bessel's inequality we get, taking the sum over eigenfunctions of H_θ attached to j,

$$1 = \|\delta_j\|^2 \geq \sum_u |\langle u, \delta_j \rangle|^2 = \sum_u |u(j)|^2.$$

So $u(j)$ attains a maximum in each $\mathbb{T}_{j,k}$ and this maximum is the image of only finitely many points.

In $\mathbb{T}_{j,1}$, define $u_1(\cdot;\theta,j)$ as the eigenfunction of H_θ with greatest value of $u(j)$. If there is more than one such function, pick the one with the greatest eigenvalue. Let $E_1(\theta,j)$ be such eigenvalue. Similarly, at each $\mathbb{T}_{j,k}$ define $u_k(\cdot;\theta,j)$ as the eigenfunction of H_θ with k$^{\text{th}}$ greatest value of $u(j)$ and if there is more then one function that attains such value at j, pick the one with the greatest eigenvalue. Call this eigenvalue $E_k(\theta,j)$. This way we construct a set $\{u_k(\cdot;\theta,j)\}_{k=1}^{N_j(\theta)}$ of the eigenvectors of H_θ attached to j such that $u_k(j;\theta,j) \geq u_{k+1}(j;\theta,j)$ and, if equality holds, then $E_k(\theta,j) > E_{k+1}(\theta,j)$. Extend E_k and u_k to all of $\mathbb{T}$ setting them to be zero in $\mathbb{T} \setminus \mathbb{T}_{j,k}$.

Our goal in this section is to show that the functions N_j, E_k and u_k are measurable in $\mathbb{T}$ (with respect to the Borel $\sigma-$algebra). We will do two simplifications: we will assume $j = 0$, which is no loss of generality since if $u(n)$ is an eigenfunction of H_θ attached to j, then $\tilde{u}(n) = u(n+j)$ is an eigenfunction of $H_{\theta - j\alpha}$ attached to 0 and translations on $\mathbb{T}$ are measurable; and instead of picking eigenfunctions u with norm 1, it will be more convinient to use $\eta(n) = u(n)/u(0)$. Since $\|\eta\| = 1/u(0)$, ordering by maximum $u(0)$ is the same as ordering by minimum $\|\eta\|$ and, since $u(n) = \eta(n)/\|\eta\|$, if we prove measurability of η we prove measurability of u. Before we prove this, one elementary statement:

Proposition 2.1. *Let a sequence $(\theta_m)_{m\in\mathbb{N}} \subset \mathbb{T}$ be such that $\lim_{m\to\infty} \theta_m = \theta$ and denote H_{θ_m} by H_m. Assume that for each m there are $\eta_m \in \ell^2$ and $E_m \in \mathbb{R}$ such that $H_m = E_m\eta_m$, $\eta_m(0) = 1$ and $\sup_{m\in\mathbb{N}} \|\eta_m\| < \infty$. Then there exist a subsequence m_i, a vector η and a real number E such that*

1. $\lim_{i\to\infty} \eta_{m_i}(n) = \eta(n)$ *for every* $n \in \mathbb{Z}$;
2. $\lim_{i\to\infty} E_{m_i} = E$;
3. $H_\theta\eta = E\eta$;
4. $\eta(0) = 1$;
5. $\|\eta\| \leq \sup_{m\in\mathbb{N}} \|\eta_m\|$.

Proof. The sequence $(\eta_m)_{m\in\mathbb{Z}}$ is bounded and the unit ball of ℓ^2 is compact and metrizable in the weak topology. Furthermore, since E_m is an eigenvalue of H_m, we have $|E_m| < 2+2\lambda$ for all m. Thus, we can choose subsequences η_{m_i} and E_{m_i} such that 1 and 2 holds. Taking the pointwise limit of $H_m = E_m\eta_m$ we obtain 3. Items 4 and 5 are immediate. □

Theorem 2.2. *For each fixed j, k and n, the functions $u_k(n;\theta,j)$ and $E_k(\theta,j)$ are measurable in $\mathbb{T}$.*

Proof. Let $E_m(\theta) = E_m(\theta, 0)$ and $\eta_m(n;\theta) = \dfrac{u_m(n;\theta,0)}{u_m(0;\theta,0)}$. Fix a pair of positive integers k, l. By Bessel's inequality $\sum_m \dfrac{1}{\|\eta_m\|^2} \leq 1$, so there are at most k^2 natural m such that $\|\eta_m\| \leq k$, let then $M_{k,l}(\theta)$ be the maximum number of m's such that $\|\eta_m\| \leq k$ and $|E_m - E_{m'}| > 2^{-l}$ for every $m' \neq m$. We claim that $S = \{\theta \in \mathbb{T} \,|\, M_{k,l}(\theta) \geq p\}$ is closed.

Indeed, if $(\theta_s) \subset S$ is such that $\theta_s \to \theta$, by Proposition 2.1 we can choose p different η's and E's such that $H_\theta \eta = E\eta$ and $\|\eta\| \leq k$. Besides, each E clearly still satisfies $|E - E'| \geq 2^{-l}$, so $M_{k,l}(\theta) \geq p$, that is $\theta \in S$. Since S is closed, $M_{k,l}$ is measurable. Now let $M_k(\theta)$ be the number of m's such that $\|\eta_m\| \leq k$. Since the spectrum is simple, we have $M_k(\theta) = \max_l M_{k,l}(\theta)$, so M_k is also measurable.

Let $S_{k,l,p} = \{\theta \in \mathbb{T} \,|\, M_{k,l}(\theta) = M_k(\theta) = p\}$ be the set of θ's such that H_θ has p eigenvalues with $\|\eta_m\| \leq k$ so that $|E_m - E_{m'}| > 2^{-p}$. Order them in a way that $\|\eta_1\| \leq \cdots \leq \|\eta_p\|$ and if $\|\eta_i\| = \|\eta_{i+1}\|$ then $E_i > E_{i+1}$. If $(\theta_m) \subset S_{k,l,p}$ converges to $\theta \in S_{k,l,p}$, then $E_i^{(m)}$ has different limit points for each i, since $|E_i^{(m)} - E_j^{(m)}| > 2^{-l}$. By Proposition 2.1, such limit points must be eigenvalues of H_θ with eigenvectors η_i such that $\|\eta_i\| \leq k$. Since the operator H_θ cannot have more than p such eigenvalues, then the limit points of $E_i^{(m)}$ and $\eta_i^{(m)}$ are all the p eigenvalues and eigenvectors of H_θ and respect the previous order of η_i. Each E_i and η_i is then continuous on $S_{k,l,p}$ (and therefore measurable) for every k, l and p. Therefore, they are measurable in $\mathbb{T}$.

□

2.2 Operator duality

Let $\mathcal{H}$ be the set of functions $\varphi : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that

$$\sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{T}} |\varphi(\theta, n)|^2 \, d\nu(\theta) \right) < \infty.$$

That is, $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{Z})$.

We define the linear operators Q_λ and U acting on $\mathcal{H}$ by

$$(Q_\lambda \varphi)(\theta, n) = \varphi(\theta, n+1) + \varphi(\theta, n-1) + 2\lambda \cos(2\pi(\alpha n + \theta))\varphi(\theta, n), \tag{2.1}$$

and

$$(U\varphi)(\theta, n) = \widehat{\varphi}(n, \theta + \alpha n), \tag{2.2}$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ. In particular, if $\varphi \in L^1(\mathbb{T} \times \mathbb{Z}) \cap L^2(\mathbb{T} \times \mathbb{Z})$,

$$(U\varphi)(\theta, n) = \sum_{m\in\mathbb{Z}} \left(e^{-2i\pi m(\theta+\alpha n)} \int_{\mathbb{T}} e^{-2i\pi n\phi} \varphi(\phi, m) d\nu(\phi) \right). \tag{2.3}$$

It is direct from its definition that U is unitary. We can now state a version of Aubry duality:

Theorem 2.3.

$$U^{-1} Q_\lambda U = \lambda Q_{\lambda^{-1}}. \tag{2.4}$$

Proof. Let T be given by $(T\varphi)(\theta, n) = \varphi(\theta, n+1)$ and M be given by $(M\varphi)(\theta, n) = e^{-2i\pi(\theta+n\alpha)}\varphi(\theta, n)$. By the properties of the Fourier transform, we have that $(U^{-1}T^kU\varphi)(\theta, n) = e^{-2i\pi k(\alpha n+\theta)}\varphi(\theta, n)$ and $(U^{-1}M^kU\varphi)(\theta, n) = \varphi(\theta, n+k)$ for every $k \in \mathbb{Z}$. Thus

$$\begin{aligned}
&(U^{-1}Q_\lambda U\varphi)(\theta, n) = (U^{-1}(T + T^{-1} + \lambda M + \lambda M^{-1})U\varphi)(\theta, n) \\
&= e^{-2i\pi(\alpha n+\theta)}\varphi(\theta, n) + e^{2i\pi(\alpha n+\theta)}\varphi(\theta, n) + \lambda\varphi(\theta, n+1) + \varphi(\theta, n-1) \\
&= \lambda\left(\frac{2}{\lambda}\cos(2\pi(\alpha n+\theta))\varphi(\theta, n) + \varphi(\theta, n+1) + \varphi(\theta, n-1)\right) = \lambda(Q_{\lambda^{-1}}\varphi)(\theta, n).
\end{aligned}$$

□

Corollary 2.3.1. *For every irrational α, we have $\sigma(H_{\lambda,\alpha,\theta}) = \lambda\sigma(H_{\lambda^{-1},\alpha,\theta})$.*

Proof. Since $\sigma(H_{\lambda,\alpha,\theta})$ is independent of θ, the spectrum of $H_{\lambda,\alpha,\theta}$ and the spectrum of Q_λ coincide. So $\sigma(H_{\lambda,\alpha,\theta}) = \sigma(Q_\lambda) = \lambda\sigma(Q_{\lambda^{-1}}) = \lambda\sigma(H_{\lambda^{-1},\alpha,\theta})$. □

Define now, for $l \in \mathbb{Z}$, the operator $(S_l\varphi)(\theta, n) = \varphi(\theta + \alpha l, n - l)$. By Fourier transform properties, we get that $(US_l\varphi)(\theta, n) = e^{-2\pi il\theta}(U\varphi)(\theta, n)$.

Proposition 2.4. *Let $\varphi \in \mathcal{H}$ be such that $l \neq 0 \implies \langle S_l\varphi, \varphi\rangle = 0$. If $g \in L^1(\mathbb{T})$ is defined by*

$$g(\theta) = \sum_{n\in\mathbb{Z}} |(U\varphi)(\theta, n)|^2,$$

then g is constant almost everywhere.

Proof. We have, for $l \in \mathbb{Z}^*$,

$$\int_{\mathbb{T}} e^{2\pi il\theta} g(\theta) d\nu(\theta) = \langle U\varphi, e^{2\pi il\theta}U\varphi\rangle = \langle U\varphi, US_{-l}\varphi\rangle = \langle\varphi, S_{-l}\varphi\rangle = 0.$$

But the span of $\{e^{2\pi il\theta}\}_{l\in\mathbb{Z}}$ is dense in $L^\infty(\mathbb{T})$ in the weak-$*$ topology, so g is constant. □

Theorem 2.5. *Fix positive λ and irrational α. Let $u_k(\cdot;\theta,j)$ be the measurable function defined in the previous section for $H_{\lambda^{-1},\alpha,\theta}$. Let $f \in L^2(\mathbb{T})$ and define, for fixed j and k, $\varphi(\theta,n) = f(\theta)u_k(n;\theta,j)$. For each $\phi \in \mathbb{T}$, define $\psi_\phi \in \ell^2$ by $\psi_\phi(n) = (U\varphi)(\phi,n)$ and let $d\mu_\phi$ be the spectral measure for the operator $H_{\lambda,\alpha,\theta}$ and the vector ψ_ϕ. Then $d\mu_\phi$ is almost everywhere $\phi-$independent.*

Proof. We have that $(S_l\varphi)(\theta,n) = f(\theta+l\alpha)u_k(n-l;\theta+l\alpha,j)$. But by the discussion of the last section, $u_k(n-l;\theta+l\alpha,j) = u_k(n;\theta,j+l)$, so $u_k(n-l;\theta+l\alpha,j)$ and $u_k(n;\theta,j)$ can be seen as eigenfunctions of the same operator, and they are linearly independent if $l \neq 0$, since they are attached to different points. So, for $l \neq 0$

$$\sum_{n\in\mathbb{Z}} \overline{u_k(n-l;\theta+l\alpha,j)}u_k(n;\theta,j) = 0 \implies \langle S_l\varphi,\varphi\rangle_{\mathcal{H}} = 0.$$

Furthermore, for every $h \in C(\mathbb{T})$ we have $h(\lambda Q_{\lambda^{-1}}\varphi)(\theta,n) = h(\lambda E_k(\theta,j))\varphi(\theta,n)$ by definition of φ. So for every $l \neq 0$ and every continuous h we have that $h(\lambda Q_{\lambda^{-1}})\varphi$ is orthogonal to $S_l\varphi$. By the same reasoning of the prove of Proposition 2.4, we get that

$$\sum_{n\in\mathbb{Z}} \overline{(U\varphi)(\phi,n)}(Uh(\lambda Q_{\lambda^{-1}}\varphi))(\phi,n) \tag{2.5}$$

is independent of ϕ.

But by Theorem 2.3, $Uh(\lambda Q_{\lambda^{-1}}) = h(Q_\lambda)U$, so from (2.5) we get that

$$\sum_{n\in\mathbb{Z}} \overline{(U\varphi)(\phi,n)}(h(Q_\lambda)U\varphi)(\phi,n) = \langle U\varphi, h(Q_\lambda)U\varphi\rangle_{\ell^2} = \int_{\mathbb{R}} h(E)d\mu_\phi(E)$$

is independent of ϕ for every continuous h. Since the set of continuous functions is dense in the set of integrable functions, $d\mu_\phi$ is almost everywhere $\phi-$constant. □

2.3 Mutual singularity

We will use two results from Spectral Theory.

Theorem 2.6. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a bounded real sequence and let the operator H_ω be defined by $(H_\omega u)(n) = u(n+1) + u(n-1) + \omega_n u(n)$. Let $G_\omega(n,m;z)$ be the matrix elements of $(H_\omega - z)^{-1}$. If $d\mu_\omega^s$ is the singular part of a spectral measure for H_ω, then $d\mu_\omega^s$ is supported by*

$$\mathfrak{G}_\omega = \left\{ E \in \mathbb{R} \,\middle|\, \lim_{\varepsilon\to 0^+} Im(G_\omega(0,0;E+i\varepsilon)) + Im(G_\omega(1,1;E+i\varepsilon)) = \infty \right\}.$$

Theorem 2.7. *Fix positive λ and irrational α. In the notation of Theorem 2.6, let ω be the sequence $(2\lambda\cos(2\pi(n\alpha+\theta)))_{n\in\mathbb{Z}}$. Then $\nu(\{\theta\in\mathbb{T}\,|\,E\in\mathfrak{G}_\omega\})=0$ for every $E\in\mathbb{R}$.*

Proof for the former can be found in Appendix B of [33]; for the latter, it can be found in [17].

Now we get to the principal result of this section:

Theorem 2.8. *Fix positive λ and irrational α. Let μ_θ be a spectral measure for $H_{\lambda,\alpha,\theta}$ and μ_θ^s be its singular part. Then for almost every θ and θ', we have $\mu_\theta^s\perp\mu_{\theta'}^s$.*

Proof. We have

$$\int_{\mathbb{T}}\mu_\theta(\mathfrak{G}_{\theta'})d\nu(\theta')=\int_{\mathbb{T}}\left(\int_{\mathbb{R}}\mathbb{1}_{\mathfrak{G}_{\theta'}}(E)\,d\mu_\theta(E)\right)d\nu(\theta')$$
$$=\int_{\mathbb{R}}\left(\int_{\mathbb{T}}\mathbb{1}_{\mathfrak{G}_{\theta'}}(E)d\nu(\theta')\right)\mu_\theta(E)=\int_{\mathbb{R}}\nu(\{\theta'\in\mathbb{T}\,|\,E\in\mathfrak{G}_{\theta'}\})\mu_\theta(E).$$

So by Theorem 2.7, $\mu_\theta(\mathfrak{G}_{\theta'})=0$ for almost every θ', which implies $d\mu_\theta\perp d\mu_{\theta'}^s$ for almost every θ'. □

2.4 Relation of dual spectra

Theorem 2.9. *Fix positive λ and irrational α. Let $u_k(\cdot;\theta,j)$ be the measurabe function defined in the section 2.1 for $H_{\lambda^{-1},\alpha,\theta}$. Let $f\in L^2(\mathbb{T})$ and define, for fixed j and k, $\varphi(\theta,n)=f(\theta)u_k(n;\theta,j)$. For each $\phi\in\mathbb{T}$, define $\psi_\phi\in\ell^2$ by $\psi_\phi(n)=(U\varphi)(\phi,n)$ and let $d\mu_\phi$ be the spectral measure for the operator $H_{\lambda,\alpha,\phi}$ and the vector ψ_ϕ. Then $d\mu_\phi$ is absolutely continuous for almost every ϕ.*

Proof. By Theorem 2.5, $d\mu_\phi$ is independent of ϕ for almost every ϕ. But by Theorem 2.8 there is also a set of full measure such that $d\mu_\phi^s\perp d\mu_{\phi'}^s$ for $\phi\neq\phi'$. But the only way both conditions can be true is if $d\mu_\phi^s\equiv 0$ for almost every ϕ. □

Theorem 2.10. *Fix positive λ and irrational α. If $H_{\lambda^{-1},\alpha,\theta}$ has point spectrum for θ in a subset of $\mathbb{T}$ of positive measure, then $H_{\lambda,\alpha,\theta}$ has some absolutely continuous spectrum for almost every θ.*

Proof. By construction, $\{u_k(\cdot;\theta,j)\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ spans the point subspace of ℓ^2 for $H_{\lambda^{-1},\alpha,\theta}$, so if there is some point spectrum, there is some j such that $u_1(\cdot;\cdot,j)$ is a non zero element of $\mathcal{H}$. By Theorem 2.9 we have that $d\mu_\phi$, which is a spectral measure for $H_{\lambda,\alpha,\phi}$, is purely absolutely continuous for almost every ϕ. Now

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} d\mu_\phi(E) \right) d\nu(\phi) = \int_{\mathbb{T}} \left(\sum_{n \in \mathbb{Z}} |(Uu_1)(n;\phi,j)|^2 \right) d\nu(\phi) = \|Uu_1\|_{\mathcal{H}} = \|u_1\|_{\mathcal{H}} > 0.$$

So $\int_{\mathbb{R}} d\mu_\phi(E) \neq 0$ for ϕ in a subset of $\mathbb{T}$ of positive measure. But $d\mu_\phi$ is almost everywhere independent of ϕ, so there is some absolutely continuous spectrum for $H_{\lambda,\alpha,\theta}$ for almost every θ. □

Theorem 2.11. *If α is irrational and λ is such that $H_{\lambda^{-1},\alpha,\theta}$ has only point spectrum for almost every θ, then $H_{\lambda,\alpha,\theta}$ has only absolutely continuous spectrum for almost every θ.*

Proof. Let $\{f_m\}$ be an orthonormal basis for $L^2(\mathbb{T})$. By hypothesis, $\{u_k(\cdot;\theta,j)\}_{j,k}$ is an orthonormal basis of ℓ^2 if we only allow j,k such that $u_k(\cdot;\theta,j) \neq 0$. Let $\varphi_{j,k,m}(\theta,n) = f_m(\theta)u_k(n;\theta,j)$, so that $\{\varphi_{j,k,m}\}_{j,k,m}$ is a complete orthogonal set of $\mathcal{H}$. Since U is unitary, $\{U\varphi_{j,k,m}\}_{j,k,m}$ also is a complete orthogonal set of $\mathcal{H}$, so for almost every ϕ, $\{(U\varphi_{j,k,m}(\phi,\cdot)\}_{j,k,m}$ is a complete set of ℓ^2. But by Theorem 2.9, these vectors are in the absolutely continuous subspace of of ℓ^2 for $H_{\lambda,\alpha,\phi}$, which completes the proof. □

3 Supercritical case

Now we have gathered the duality results, in this chapter our aim is to show that in the supercritical case, that is, for $\lambda > 1$, the almost Mathieu operator has only pure point spectrum for almost every frequency (diophantine α)[1] and almost every phase. Before we talk about how this chapter is organized, we define the set of phases for which this classification of the spectrum might not hold, show that it is indeed a set of measure zero and then state the main theorem of this chapter.

Definition 3.1. *Given diophantine α; $r \in \mathbb{N}$ such that $|q\alpha - p| > cq^{-r}$ for some $c > 0$ and every $(p, q) \in \mathbb{Z} \times \mathbb{N}$; and $n \in \mathbb{N}$ let*

$$\Theta_n = \left\{ \theta \in \mathbb{R} \;\middle|\; \left| \sin\left(2\pi\left(\theta + \frac{n}{2}\alpha\right)\right)\right| < \exp\left(-n^{1/(2r)}\right) \right\}. \tag{3.1}$$

We then define the set of ***resonant phases*** *by*

$$\Theta = \limsup_n \Theta_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Theta_k.$$

Proposition 3.2. *The set of resonant phases is a set with measure zero.*

Proof. By periodicity, it is sufficient to show $\nu\left(\Theta \cap \left[-\frac{1}{4}, \frac{1}{4}\right]\right) = 0$.

For fixed n we have that if $\theta \in \Theta_n \cap \left[-\frac{1}{4} - \frac{n}{2}\alpha, \frac{1}{4} - \frac{n}{2}\alpha\right]$ then $\left|2\pi\left(\theta + \frac{n}{2}\alpha\right)\right| \leq \frac{\pi}{2}$. Thus we can use the estimate $|\sin x| \geq \frac{|x|}{2}$, which gives us

$$\exp\left(-n^{1/(2r)}\right) > \left|\sin\left(2\pi\left(\theta + \frac{n}{2}\alpha\right)\right)\right| \geq \pi\left|\theta + \frac{n}{2}\alpha\right|.$$

$$\text{So } \Theta_n \cap \left[-\frac{1}{4} - \frac{n}{2}\alpha, \frac{1}{4} - \frac{n}{2}\alpha\right] \subset \left(-\frac{n}{2}\alpha - \frac{\exp\left(-n^{1/(2r)}\right)}{\pi}, -\frac{n}{2}\alpha + \frac{\exp\left(-n^{1/(2r)}\right)}{\pi}\right).$$

By periodicity, we get $\nu\left(\Theta_n \cap \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \leq \frac{2\exp\left(-n^{1/(2r)}\right)}{\pi}$. Since $\sum_{n=1}^{\infty} \exp\left(-n^{1/(2r)}\right)$ converges, the Borel-Cantelli lemma gives us $\nu\left(\Theta \cap \left[-\frac{1}{4}, \frac{1}{4}\right]\right) = 0$.

□

We now state the main theorem of this chapter, which will be proved in the second section:

[1] See Appendix D.

Theorem 3.3. *Let $\lambda > 1$, α diophantine and θ a non-resonant phase. Then the spectrum of the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ is pure point.*

By Theorem 2.11 we have

Corollary 3.3.1. *Let $\lambda < 1$ and α diophantine. Then the spectrum of the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ is purely absolutely continuous for almost every θ.*

This chapter was sectioned in the following way: in the first section of this chapter we deal with some important results that will be used to prove Theorem 3.3: in the first subsection of section 3.1, we define some dynamical quantities related to the equation $H_{\lambda,\alpha,\theta}u = Eu$ for real E, such as the Lyapunov exponent. We will allow generalized solutions, that is, we only require $u \in \mathbb{C}^{\mathbb{Z}}$, not $u \in \ell^2$. We will also define regular and singular points of a solution, which is linked to the behavior of the Green function on intervals where such points lie. The main results of this subsection are the positivity of the Lyapunov exponent for the supercritical case, and that if a solution of $Hu = Eu$ is such that $u(n) \neq 0$, then n is a singular point for suitable parameters.

In the second subsection of section 3.1, we define normal functions and uniformly distributed points and use these definitions to get upper bounds on the growth of dynamical parameters introduced in the first subsection. These bounds use the arithmetical properties of diophantine frequencies and non-resonant phases.

In section 3.2, we use such bounds to show that singular points, if not very close to each other, must be far apart. Using then the singularity of points in which the solutions of $H_{\lambda,\alpha,\theta}u = Eu$ don't vanish, and the positivity of the Lyapunov exponent, we prove that for a dense set of E in the the spectrum the associated generalized solution of $Hu = Eu$ is indeed an eigenfunction of H, which allows us to finally conclude that $H_{\lambda,\alpha,\theta}$ has pure point spectrum for $\lambda > 1$, diophantine α and non-resonant θ.

In the the third section, we show that it is not possible to change from "almost every frequency and almost every phase" to "every frequency and every phase": in subsection 3.3.1 we show that there is a dense G_δ of frequencies such that for every phase there can never be point spectrum; and in subsection 3.3.2 we show that for every frequency, there is a dense G_δ of frequencies for which there can never be point spectrum.

For sections 3.1 and 3.2 our main reference is [24]. In subsection 3.3.1 we use [8, 32] and in section 3.3.2 we use [26]. To avoid cumbersome notation, we shall omit the dependence on the parameters λ, α, θ and E throughout most of the chapter when there is no risk of confusion.

3.1 Preliminaries

3.1.1 The Dynamics of the eigenvalue equation

Let $T(\cdot,\theta)$ be the transfer matrix[2] of the eigenvalue equation $H_{\lambda,\alpha,\theta}u = Eu$. We define the **Lyapunov exponent** $\gamma(E)$ by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n}\int_0^1 \ln\|T(n,\theta)\|d\theta = \inf_{n\in\mathbb{N}} \frac{1}{n}\int_0^1 \ln\|T(n,\theta)\|d\nu(\theta). \tag{3.2}$$

The limit exists and the second equality holds by the subadditive ergodic theorem. By the limit definition, it is clear that equivalent norms lead to the same $\gamma(E)$. Since all matrix norms are equivalent, when necessary we will use the Hilbert-Schimdt norm for convenience.

Proposition 3.4. *For every real E, we have $\gamma(E) \geq \ln\lambda$. In particular, for the supercritical case we always have $\gamma(E) > 0$.*

Proof. The proof uses a subharmonicity argument due to Herman [22].

For $z \in \mathbb{C}$ and $n \geq 0$, let $M(n,z) = \begin{pmatrix} Ez - \lambda\left(e^{2\pi i\alpha n}z^2 + e^{-2\pi i\alpha n}\right) & -z \\ z & 0 \end{pmatrix}$ and define $N_n(z)$ recursively on n by

$$N_n(z) = \begin{cases} M(0,z), & \text{if } n = 0 \\ M(n-1,z)N_{n-1}(z) & \text{if } n > 0. \end{cases}$$

Since N_n is entire for every n, $f_n : \mathbb{C} \to \mathbb{C}$ given by $f_n(z) = \ln\|N_n(z)\|$ is subharmonic. So

$$n\ln\lambda = \|N_n(0)\| \leq \int_0^1 \ln\|N_n(e^{2\pi i\theta})\|d\nu(\theta).$$

If $\theta \in \mathbb{R}$ and $A(n,\theta) = \begin{pmatrix} E - 2\lambda\left(\cos\left(2\pi(n\alpha+\theta)\right)\right) & -1 \\ 1 & 0 \end{pmatrix}$, then $M(n,e^{2\pi i\theta}) = e^{2\pi i\theta}A(\theta+n\alpha)$, so $N_n(e^{2\pi i\theta}) = e^{2\pi in\theta}T_n(\theta)$. This gives us

$$\gamma(E) = \lim_{n\to\infty}\frac{1}{n}\int_0^1 \ln\|T_n(\theta)\|d\nu(\theta) = \lim_{n\to\infty}\frac{1}{n}\int_0^1 \ln\|N_n(e^{2\pi i\theta})\|d\nu(\theta) \geq \ln\lambda.$$

□

Remark 3.5. *Although this will not be used here, in [13] Bourgain and Jitomirskaya proved that $\gamma(E) = \max\{0, \ln\lambda\}$ for every E in the spectrum.*

[2] See Appendix C.

Given integers $n_1 < n_2$, let $H_{[n_1,n_2]}$ be the restriction of H to the interval $[n_1, n_2] \subset \mathbb{Z}$. For $m, n \in [n_1, n_2]$, we define the Green function $G_{[n_1,n_2]}(m, n)$ as the matrix elements of the operator $(H_{[n_1,n_2]} - E)^{-1}$. Based on the behavior of the Green function, we have the following definitions:

Definition 3.6. *Let $m \in \mathbb{Z}$, $k \in \mathbb{N}$ and $\beta > 0$. Then m is called a (β, k)-**regular point** if there is an interval $[n_1, n_2] = [n_1, n_1 + k - 1]$ such that for $i \in \{1, 2\}$ we have $|G_{[n_1,n_2]}(m, n_i)| < e^{-\beta|m-n_i|}$ and $|m - n_i| \geq \frac{1}{40}k$. Otherwise, m is called (β, k)-**singular**.*

Proposition 3.7. *Let u be a solution of $Hu = Eu$ such that, for some $B > 0$, we have $|u(n)| \leq B(1 + |n|)$ for all $n \in \mathbb{Z}$. If $m \in \mathbb{Z}$ is such that $u(m) \neq 0$, then for every $\beta > 0$ there is $k_1 = k_1(E, \beta, \theta, m)$ such that m is $(\beta, k)-$singular for $k > k_1$.*

Proof. We know that for $n \in [n_1, n_2]$, we can get the value $u(n)$ for a formal solution of $Hu = Eu$ using the boundary values by

$$u(n) = -G_{[n_1,n_2]}(n, n_1)u(n_1 - 1) - G_{[n_1,n_2]}(n, n_2)u(n_2 + 1). \tag{3.3}$$

If the proposition does not hold, there are infinite $k \in \mathbb{N}$ such that m is $(\beta, k)-$regular. So, for infinite naturals k,

$$\begin{aligned}
&|u(m)| \leq |G_{[n_1,n_2]}(m, n_1)||u(n_1 - 1)| + |G_{[n_1,n_2]}(m, n_2)||u(n_2 + 1)| \\
&< e^{-\beta k/40} \left(|u(n_1 - 1)| + |u(n_2 + 1)|\right) \leq Be^{-\beta k/40} \left(|n_1 - 1| + |n_2 + 1| + 2\right) \\
&\leq Be^{-\beta k/40} \left(|n_1 - m| + |n_2 - m| + 2|m| + 4\right) \leq Be^{-\beta k/40} \left(\frac{k}{20} + 4 + 2|m|\right).
\end{aligned}$$

This contradicts $u(m) \neq 0$.

□

We then define for natural n the function $P_n : \mathbb{R} \to \mathbb{R}$ by $P_n(\theta) = \det\left(E - H_{[0,n-1]}\right)$. For convenience, we also set $P_0(\theta) = 1$ and $P_{-1}(\theta) = 0$. With this convention, it is possible to show that for every $n \geq 1$ the transfer matrix can be written as

$$T_n(\theta) = \begin{pmatrix} P_n(\theta) & -P_{n-1}(\theta + \alpha) \\ P_{n-1}(\theta) & -P_{n-2}(\theta + \alpha) \end{pmatrix}. \tag{3.4}$$

Furthermore, by Cramer's rule we get

$$|G_{[n_1,n_2]}(n_1, m)| = \left|\frac{P_{n_2-m}(\theta + (m + 1)\alpha)}{P_k(\theta + n_1\alpha)}\right|, \tag{3.5}$$

and

$$|G_{[n_1,n_2]}(m, n_2)| = \left|\frac{P_{m-n_1}(\theta + n_1\alpha)}{P_k(\theta + n_1\alpha)}\right|, \tag{3.6}$$

where $k = n_2 - n_1 + 1$.

We will now look further into properties of the functions P_n for the proofs in the section 3.2 rely entirely on bounds of $|P_n(\theta)|$.

Proposition 3.8. *Let* $\mathcal{K} = \left\{ k \in \mathbb{N} \,\middle|\, \exists \theta \in [0,1] \text{ such that } |P_n(\theta)| \geq \frac{1}{2} e^{k\gamma(E)} \right\}$. *If* $\lambda > 1$, *for every* $n \in \mathbb{N}$ *we have* $\{n, n+1, n+2\} \cap \mathcal{K} \neq \varnothing$.

Proof. Assume, for the sake of contradiction, that the proposition is false. Then there exists a natural n such that $\{n, n+1, n+2\} \cap \mathcal{K} = \varnothing$. We then get

$$\begin{aligned}
&\|T_{n+2}(\theta)\|^2 = |P_n(\theta+\alpha)|^2 + |P_{n+1}(\theta+\alpha)|^2 + |P_{n+1}(\theta)|^2 + |P_{n+2}(\theta)|^2 \\
&< \frac{1}{4} e^{2n\gamma(E)} + \frac{1}{2} e^{2(n+1)\gamma(E)} + \frac{1}{4} e^{2(n+2)\gamma(E)} = \frac{1}{4} e^{2(n+2)\gamma(E)} \left(e^{-2\gamma(E)} + 1 \right)^2 .
\end{aligned}$$

Since $\gamma(E) > 0$, we then have

$$\begin{aligned}
&\frac{1}{n+2} \ln \|T_{n+2}(\theta)\| < \frac{1}{n+2} \ln \|(e^{(n+2)\gamma(E)} \left(e^{-2\gamma(E)} + 1 \right))/2\| \\
&= \gamma(E) - \ln 2 + \ln \left(e^{-2\gamma(E)} + 1 \right) < \gamma(E).
\end{aligned}$$

But this is absurd by the infimum definition in (3.2).

□

Lemma 3.9. *Given the almost Mathieu operator* $H_{\lambda,\alpha,\theta}$, *for every* $E \in \mathbb{R}$ *and every* $\varepsilon > 0$ *there is* $k_2 = k_2(E, \alpha, \varepsilon)$ *such that* $|P_n(\theta)| < e^{n(\gamma(E)+\varepsilon)}$ *for every* $n > k_2$ *and every* θ.

Proof. We will use an ergodic theorem instead of the proof in [24]. By the Subadditive Ergodic Theorem for Uniquely Ergodic Systems (see [19]), the inequality $\limsup_n \frac{1}{n} \ln \|T_n(\theta)\| \leq \gamma(E)$ holds uniformly in $\mathbb{T}$, i.e., there is $k_2 = k_2(E, \alpha, \varepsilon)$ such that $n > k_2$ implies $\frac{1}{n} \ln \|T_n(\theta)\| < \gamma(E) + \varepsilon$ for every θ. Since by (3.4) we have $|P_n(\theta)| \leq \|T_n(\theta)\|$, we get that, for every θ,

$$n > k_2 \implies \frac{1}{n} \ln \|T_n(\theta)\| < \gamma(E) + \varepsilon \implies |P_n(\theta)| < e^{n(\gamma(E)+\varepsilon)}.$$

□

Lemma 3.10. *Let* $m \in \mathbb{Z}$ *be* $(\gamma(E) - \varepsilon, k)$*-singular point with* $0 < \varepsilon < \gamma(E)$ *and* $k > 8k_2(E, \alpha, \varepsilon/6) + 6$. *If* $n \in \mathbb{Z}$ *satisfies* $m - 7k/8 \leq n \leq m - k/8$, *then* $|P_k(\theta + n\alpha)| \leq e^{k(\gamma(E)-\varepsilon/12)}$.

Proof. Let $[n_1, n_2] = [n, n+k-1]$. Since k_2 can be chosen greater than 10, we get the inequalities

$$|m - n_1| = m - n \geq \frac{k}{8} > \frac{k}{40};$$

and

$$|m-n_2| = n_2 - m = n - m + k - 1 \geq -\frac{7k}{8} + k - 1 \geq \frac{k}{8} - 1 \geq \frac{k}{40}.$$

Therefore, by $(\gamma(E)-\varepsilon, k)$-singularity of m, we have $|G_{[n_1,n_2]}(m,n_2)| \geq e^{-|m-n_2|(\gamma(E)-\varepsilon)}$. Furthermore, since $|m-n_1| \geq \frac{k}{8} > k_2(E,\alpha,\varepsilon/6)$ we have $|P_{m-n_1}(\theta+n_1\alpha)| < e^{|m-n_1|(\gamma(E)+\varepsilon/6)}$ by lemma 3.9. From equation (3.6) it follows that

$$|P_k(\theta+n\alpha)| = \frac{|P_{m-n}(\theta+n\alpha)|}{|G_{[n_1,n_2]}(m,n_2)|} \leq e^{|m-n_1|(\gamma(E)+\varepsilon/6)} e^{|m-n_2|(\gamma(E)-\varepsilon)}$$
$$= e^{(|m-n_1|+|m-n_2|)\gamma(E)-(|m-n_2|-|m-n_1|/6)\varepsilon} = e^{(k-1)\gamma(E)-(|m-n_2|-|m-n_1|/6)\varepsilon}.$$

Using now that $|m-n_2| \geq \frac{k}{8} - 1$ and $|m-n_1| \leq \frac{7k}{8}$ we get $|m-n_2| - \frac{|m-n_1|}{6} \geq \frac{k}{4} - \frac{7k}{48} = \frac{5k}{48} \geq \frac{k}{12}$. We then conclude

$$|P_k(\theta+n\alpha)| \leq e^{(k-1)\gamma(E)-(|m-n_2|-|m-n_1|/6)\varepsilon} \leq e^{(k-1)\gamma(E)-5k\varepsilon/48} \leq e^{k(\gamma(E)-\varepsilon/12)}.$$

□

Before we end this subsection, let's discuss one last aspect of P_n. It is clear from the matrix representation of the operator $H_{[0,n-1]}$ in the canonical basis that, using trigonometric identities, we can write P_n as a polynomial in $\cos(2\pi\theta)$, say $P_n(\theta) = \sum_{j=0}^{n} a_j \cos^j(2\pi\theta)$. Define the polynomial Q_n by $Q_n(z) = \sum_{j=0}^{n} a_j z^j$, that is, $P_n(\theta) = Q_n(\cos(2\pi\theta))$. Assume that m_1 and m_2 are $(\gamma(E)-\varepsilon, k)$-singular points with $m_1 < m_2$ and let $d = m_2 - m_1$ and $n_i = m_i - \left\lfloor \frac{7k}{8} \right\rfloor \geq m_i - \frac{7k}{8}$. Define for $j = 0, 1, \ldots, k$

$$\theta_j = \begin{cases} \theta + (n_1 + j)\alpha, & \text{if } 0 \leq j \leq \left\lfloor \frac{k+1}{2} \right\rfloor - 1 \\ \theta + \left(n_2 + j - \left\lfloor \frac{k+1}{2} \right\rfloor\right)\alpha, & \text{if } \left\lfloor \frac{k+1}{2} \right\rfloor \leq j \leq k. \end{cases} \tag{3.7}$$

If $d > \frac{k+1}{2}$ we can guarantee that the mapping $j \mapsto \theta_j$ is injective and therefore use Lagrange interpolation to write

$$|Q_k(z)| = \left| \sum_{j=0}^{k} P_k(\theta_j) \frac{\prod_{\substack{l=0,\ldots,k \\ l\neq j}} z - \cos(2\pi\theta_l)}{\prod_{\substack{l=0,\ldots,k \\ l\neq j}} \cos(2\pi\theta_j) - \cos(2\pi\theta_l)} \right|. \tag{3.8}$$

3.1.2 Normal functions and uniformly distributed points

Definition 3.11. *We will say that $f \in C([a,b])$ is* ***normal*** *if there is p such that for every $y \in \mathbb{R}$ the cardinality of $f^{-1}(\{y\})$ is not greater than p and if for every $\varepsilon > 0$ exists $\delta > 0$ such that*

$$\left| \int_{|z-f(x)|<\delta} \ln|z - f(x)| dx \right| < \frac{\varepsilon}{2}, \qquad \forall z \in [\min f, \max f]. \tag{3.9}$$

Proposition 3.12. *The function $f(x) = \cos(2\pi x)$ is normal in $[0,1]$.*

Proof. Clearly, f takes each value in $[-1,1]$ no more than 2 times. So it is left to show the integral condition. First note that, given $a \in [-1,1]$ and $x_0 \in [0,1]$ such that $f(x) = a$, using l'Hôpital we get

$$\lim_{x \to x_0} \sqrt{|x - x_0|} \ln|a - \cos(2\pi x)| = 0.$$

So there is $\xi > 0$ such that $0 < |x - x_0| < \xi \implies |\ln|a - \cos(2\pi x)|| < \dfrac{1}{\sqrt{|x - x_0|}}$.

Now given $z \in [-1,1]$ and $\varepsilon > 0$, we will treat three cases:

If $z = -1$, choose $\xi \in (0, \varepsilon^2/64)$ such that $|\ln|-1-\cos(2\pi x)|| < \dfrac{1}{\sqrt{\left|x - \frac{1}{2}\right|}}$ whenever $0 < \left|x - \dfrac{1}{2}\right| < \xi_i$. Then there is $\delta > 0$ such that $f\left(\left(\frac{1}{2} - \xi, \frac{1}{2} + \xi\right)\right) = [-1, -1+\delta)$. Since no other value in $[0,1]$ takes value in $[-1, -1+\delta)$, we have

$$\left| \int_{|-1-\cos(2\pi x)|<\delta} \ln|z - \cos(2\pi x)| dx \right| \leq \int_{\frac{1}{2}-\xi}^{\frac{1}{2}+\xi} |\ln|z - \cos(2\pi x)|| \, dx$$
$$< \int_{\frac{1}{2}-\xi}^{\frac{1}{2}-\xi} \frac{1}{\sqrt{\left|x + \frac{1}{2}\right|}} dx = 4\sqrt{\xi} < \frac{\varepsilon}{2}.$$

If $z = 1$, then choose $\xi \in (0, \varepsilon^2/64)$ so that $0 < |x| < \xi \implies |\ln|1 - \cos(2\pi x)|| < \dfrac{1}{\sqrt{|x|}}$ and that $0 < |x-1| < \xi \implies |\ln|1 - \cos(2\pi x)|| < \dfrac{1}{\sqrt{|x-1|}}$. Then there is $\delta > 0$ such that $f([0,\xi) \cup (1-\xi, 1]) = (1-\delta, 1]$. Since no other value in $[0,1]$ takes value in $(1-\delta, 1]$, we have

$$\left| \int_{|1-\cos(2\pi x)|<\delta} \ln|1 - \cos(2\pi x)| dx \right| \leq \int_0^{\xi} |\ln|1 - \cos(2\pi x)|| \, dx + \int_{1-\xi}^{1} |\ln|1 - \cos(2\pi x)|| \, dx$$
$$< \int_0^{\xi} \frac{1}{\sqrt{|x|}} dx + \int_{1-\xi}^{1} \frac{1}{\sqrt{|x-1|}} dx = 4\sqrt{\xi} < \frac{\varepsilon}{2}.$$

Finally, if $|z|<1$, let $x_1, x_2 \in [0,1]$ be such that $\{x_1, x_2\} = f^{-1}(\{z\})$ and choose $\xi \in (0, \varepsilon^2/256)$ such that $0 < |x - x_i| < \xi \implies |\ln|z - \cos(2\pi x)|| < \dfrac{1}{\sqrt{|x-x_i|}}$ and $f|_{|x-x_i|<\xi}$ is one-to-one for $i = 1, 2$. Then we can choose $\delta > 0$ such that $(z-\delta, z+\delta) \subset f((x_1-\xi, x_1+\xi) \cup (x_2-\xi, x_2+\xi))$ and no other value in $[0,1]$ takes value in $(z-\delta, z+\delta)$. Finally, we get

$$\begin{aligned}
&\left|\int_{|z-\cos(2\pi x)|<\delta} \ln|z-\cos(2\pi x)|dx\right| \\
&\leq \int_{|x-x_1|<\xi} |\ln|z-\cos(2\pi x)||\,dx + \int_{|x-x_2|<\xi} |\ln|z-\cos(2\pi x)||\,dx \\
&< \int_{|x-x_1|<\xi} \frac{1}{\sqrt{|x-x_1|}}dx + \int_{|x-x_2|<\xi} \frac{1}{\sqrt{|x-x_2|}}dx = 8\sqrt{\xi} < \frac{\varepsilon}{2}.
\end{aligned}$$

This proof can be adapted to any analytic function on a compact interval $[a,b]$ such that the cardinality of $f^{-1}(\{y\})$ is uniformly bounded. □

Definition 3.13. *Let $\phi : \mathbb{N} \to \mathbb{R}$ be a nondecreasing function such that $\lim\limits_{n\to\infty} \dfrac{\phi(n)}{n} = 0$. We say that $x_1, \ldots, x_n$ are $\phi-$**uniformly distributed** in $[a,b]$ if every $f \in C([a,b])$ satisfies the inequality*

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{\phi(n)}{n}\,Var(f).$$

Clearly, if $x_1, \ldots, x_n$ are ϕ_1-uniformly distributed and $y_1, \ldots, y_n$ are ϕ_2-uniformly distributed, then $x_1, \ldots, x_n, y_1, \ldots, y_n$ are $(\phi_1+\phi_2)-$uniformly distributed.

Lemma 3.14. *Suppose that f is normal and ϕ and ψ are real functions such that $\lim\limits_{n\to\infty} \dfrac{\phi(n)}{n} = \lim\limits_{n\to\infty} \dfrac{\psi(n)}{n} = 0$. Then for every $\varepsilon > 0$ there is N such that for every $n > N$, if $x_1, \ldots x_n$ are $\phi-$uniformy distributed in $[a,b]$ and $j_1, \ldots, j_{\psi(n)} \in \{1, \ldots, n\}$, we have the following estimates:*

$$\frac{1}{n}\sum_{\substack{i=1 \\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z-f(x_i)| \leq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)| + \varepsilon\right); \tag{3.10}$$

$$\frac{1}{n}\sum_{\substack{i=1 \\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} \ln|z-f(x_i)| \geq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)| - \varepsilon\right) + \frac{\phi(n)}{n} p \ln\left(\min_{\substack{i=1,\ldots,n \\ i\neq j_1,\ldots,j_{\psi(n)}}} |z-f(x_i)|\right). \tag{3.11}$$

Proof. Given $\varepsilon > 0$, choose $0 < \delta < 1$ such that (3.9) holds. Given $t > 0$, define the continuous function $g_t(x) = \begin{cases} \ln|x|, & \text{if } |x| > t \\ \ln t, & \text{if } |x| \leq t \end{cases}$. We then have

$$\frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n}\ln|z-f(x_i)| \leq \frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n} g_\delta\left(|z-f(x_i)|\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} g_\delta\left(|z-f(x_i)|\right) - \frac{1}{n}\sum_{i=1}^{\psi(n)} g_\delta\left(|z-f(x_{j_i})|\right)$$
$$\leq \frac{1}{b-a}\int_a^b g_\delta\left(|z-f(x)|\right)dx + \frac{\phi(n)}{n}\mathrm{Var}\left(g_\delta(|z-f(x)|)\right) - \frac{\psi(n)}{n}\ln\delta.$$

But

$$\int_a^b g_\delta\left(|z-f(x)|\right)dx = \int_{|z-f(x)|>\delta}\ln|z-f(x)|dx + \int_{|z-f(x)|\leq\delta}\ln\delta dx$$
$$= \int_a^b \ln|z-f(x)|dx - \int_{|z-f(x)|\leq\delta}\ln|z-f(x)|dx + \int_{|z-f(x)|\leq\delta}\ln\delta dx$$
$$\leq \int_a^b \ln|z-f(x)|dx - \int_{|z-f(x)|\leq\delta}\ln|z-f(x)|dx \leq \int_a^b \ln|z-f(x)|dx + \frac{\varepsilon}{2}.$$

so we get

$$\frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n}\ln|z-f(x_i)|$$
$$\leq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)|dx + \frac{\varepsilon}{2}\right) + \frac{\phi(n)}{n}\mathrm{Var}\left(g_\delta(|z-f(x)|)\right) - \frac{\psi(n)}{n}\ln\delta$$
$$\leq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)|dx + \frac{\varepsilon}{2}\right) + \frac{\phi(n)}{n}p\left(\max_{x\in[a,b]}\ln|z-f(x)| - \ln\delta\right) - \frac{\psi(n)}{n}\ln\delta.$$

Since $\lim_{n\to\infty}\frac{\phi(n)}{n}p\left(\max_{x\in[a,b]}\ln|z-f(x)| - \ln\delta\right) - \frac{\psi(n)}{n}\ln\delta = 0$, it is possible to choose N_1 such that $n > N_1 \implies \left|\frac{\phi(n)}{n}p\left(\max_{x\in[a,b]}\ln|z-f(x)| - \ln\delta\right) - \frac{\psi(n)}{n}\ln\delta\right| < \frac{\varepsilon}{2(b-a)}$, what allows us to conclude that, for $n > N_1$,

$$\frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}^{n}\ln|z-f(x_i)| \leq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)| + \varepsilon\right).$$

To get (3.11), let $c = \min_{\substack{i=1\\ i\neq j_1,\ldots,j_{\psi(n)}}}|z-f(x_i)|$. Since $ln|x| \leq g_c(x)$ for all $x \neq 0$, we have

$$\frac{1}{n}\sum_{\substack{i=1,\dots,n\\ i\neq j_1,\dots,j_{\psi(n)}}}^{n}\ln|z-f(x_i)| \geq \frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\dots,j_{\psi(n)}}}^{n} g_c\left(|z-f(x_i)|\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} g_c\left(|z-f(x_i)|\right) - \frac{1}{n}\sum_{i=1}^{\psi(n)} g_c\left(|z-f(x_{j_i})|\right)$$
$$\geq \frac{1}{b-a}\int_a^b g_c\left(|z-f(x)|\right)dx - \frac{\phi(n)}{n}\mathrm{Var}\left(g_c(|z-f(x)|)\right) - \frac{\psi(n)}{n}\max_{x\in[a,b]}\ln|z-f(x)|$$
$$\geq \frac{1}{b-a}\int_a^b g_c\left(|z-f(x)|\right)dx - \frac{\phi(n)}{n}p\left(\max_{x\in[a,b]}\ln|z-f(x)| - \ln c\right) - \frac{\psi(n)}{n}\max_{x\in[a,b]}\ln\left(|z-f(x)|\right).$$

$$\text{Choosing } N_2 \text{ such that } n > N_2 \implies \left|\frac{\max\limits_{x\in[a,b]}\ln|z-f(x)|}{n}\left(\phi(n)p+\psi(n)\right)\right| < \frac{\varepsilon}{b-a}$$

we get, for $n > N_2$

$$\frac{1}{n}\sum_{\substack{i=1\\ i\neq j_1,\dots,j_{\psi(n)}}}^{n}\ln|z-f(x_i)| \geq \frac{1}{b-a}\left(\int_a^b \ln|z-f(x)| - \varepsilon\right) + \frac{\phi(n)}{n}p\ln c.$$

Defining $N = \max\{N_1, N_2\}$ the proof is complete at last.

□

Lemma 3.15. *Let $\{x\}$ denote the fractional part of x. For every diophantine α and every $x \in \mathbb{R}$, the points $\{x\}, \{x+\alpha\}, \dots, \{x+(n-1)\alpha\}$ are $\phi-$uniformly distributed on $[0,1]$ for $\phi(n) = Cn^{1-r^{-1}}\ln n$ and $n \geq 2$, where $r \in \mathbb{N}$ such that $|q\alpha - p| > cq^{-r}$ for some $c > 0$ and every $(p,q) \in \mathbb{Z}\times\mathbb{N}$.*

Proof. Let $\frac{p_n}{q_n}$ be the sequence of continued fractions for α. By the Denjoy-Koksma inequality,for every continuous function f, every $x \in \mathbb{R}$ and every natural n we have

$$\left|\sum_{i=0}^{q_n-1} f\left(\{x+i\alpha\}\right) - q_n\int_0^1 f(\theta)\,d\theta\right| \leq \mathrm{Var}(f).$$

Given a natural k, define a nonnegative integer $n(k)$ by $q_{n(k)} \leq k < q_{n(k)+1}$. Write $k = \sum_{i=0}^{n} b_i q_i$ with $0 \leq b_{j-1}q_{j-1} \leq \sum_{i=0}^{j-1} b_i q_i < q_j$ for $1 \leq j \leq n+1$. Let $t_j = \begin{cases} 0, & \text{if } j = 0 \\ \sum_{i=0}^{j-1} b_i q_i & \text{if } 1 \leq j \leq n+1 \end{cases}$. We have

$$\left|\sum_{i=t_j}^{t_{j+1}-1} f\left(\{x+i\alpha\}\right) - b_j q_j \int_0^1 f(\theta)\,d\theta\right| = \left|\sum_{l=0}^{b_j-1}\left(\sum_{i=t_j+lq_j}^{t_j+(l+1)q_j-1} f\left(\{x+i\alpha\}\right)\right) - b_j q_j \int_0^1 f(\theta)\,d\theta\right|$$
$$= \left|\sum_{l=0}^{b_j-1}\left(\sum_{i=t_j+lq_j}^{t_j+(l+1)q_j-1} f\left(\{x+i\alpha\}\right) - q_j \int_0^1 f(\theta)\,d\theta\right)\right|$$
$$\leq \sum_{l=0}^{b_j-1}\left|\sum_{i=t_j+lq_j}^{t_j+(l+1)q_j-1} f\left(\{x+i\alpha\}\right) - q_j \int_0^1 f(\theta)\,d\theta\right| \leq b_j \mathrm{Var}(f).$$

It follows that

$$\left|\sum_{i=0}^{k-1} f\left(\{x+i\alpha\}\right) - k\int_0^1 f(\theta)\,d\theta\right| = \left|\sum_{j=0}^{n}\sum_{i=t_j}^{t_{j+1}-1} f\left(\{x+i\alpha\}\right) - \sum_{j=0}^{n} b_j q_j \int_0^1 f(\theta)\,d\theta\right|$$
$$\leq \sum_{j=0}^{n}\left|\sum_{i=t_j}^{t_{j+1}-1} f\left(\{x+i\alpha\}\right) - b_j q_j \int_0^1 f(\theta)\,d\theta\right| \leq \sum_{j=0}^{n} b_j \mathrm{Var}(f).$$

By definition, $b_j \leq \dfrac{q_{j+1}}{q_j}$. Since α is diophantine, $q_{j+1} \leq c^{-1}q_j^r$, so $b_j < \dfrac{q_{j+1}}{(cq_{j+1})^{r^{-1}}} = c^{-r^{-1}} q_j^{1-r^{-1}}$. Since $k < q_{n+1} \leq c^{-1}q_n^r \implies q_n > (ck)^{r^{-1}}$, for b_n we have a finer estimate: $b_n < \dfrac{k}{q_n} < c^{-r^{-1}}k^{1-r^{-1}}$. Therefore,

$$\left|\sum_{i=0}^{k-1} f\left(\{x+i\alpha\}\right) - k\int_0^1 f(\theta)\,d\theta\right| \leq \left(c^{-r^{-1}}k^{1-r^{-1}} + \sum_{j=0}^{n-1} c^{-r^{-1}} q_{j+1}^{1-r^{-1}}\right)\mathrm{Var}(f)$$
$$\leq c^{-r^{-1}}\left(k^{1-r^{-1}} + nq_n^{1-r^{-1}}\right)\mathrm{Var}(f).$$

Since $q_n \geq 2^{(n-1)/2} \implies n \leq \dfrac{2\ln q_n}{\ln 2} + 1$ for every n, we get

$$\left|\sum_{i=0}^{k-1} f\left(\{x+i\alpha\}\right) - k\int_0^1 f(\theta)\,d\theta\right| \leq c^{-r^{-1}}\left(k^{1-r^{-1}} + \left(\frac{2\ln q_n}{\ln 2} + 1\right) q_n^{1-r^{-1}}\right)\mathrm{Var}(f)$$
$$\leq c^{-r^{-1}}\left(2 + \frac{2\ln k}{\ln 2}\right)k^{1-r^{-1}}\mathrm{Var}(f).$$

Now taking $C = \dfrac{4c^{-r^{-1}}}{\ln 2} = c^{-r^{-1}}\max\limits_{k\geq 2}\left(\dfrac{2}{\ln k} + \dfrac{2}{\ln 2}\right)$ we get

$$\left|\sum_{i=0}^{k-1} f\left(\{x+i\alpha\}\right) - \int_0^1 f(\theta)\,d\theta\right| \leq Ck^{1-r^{-1}}\ln(k)\,\mathrm{Var}(f) = \phi(k)\mathrm{Var}(f)$$
$$\implies \left|\frac{1}{k}\sum_{i=0}^{k-1} f\left(\{x+i\alpha\}\right) - \int_0^1 f(\theta)\,d\theta\right| \leq \frac{\phi(k)}{k}\mathrm{Var}(f),$$

as we wanted. □

3.2 On the behaviour of singular points

We will now return to the discussion at the end of section 3.1. Using the results presented so far, we will show that the cossines of distinct θ_j's defined in equation (3.7) do not get too close to each other (Lemma 3.17), and that will allow us to get an exponential bound on the growth of the ratio of the products in the equation (3.8). Such bound and Lemma 3.10 will then be used to show that singular point exhibit some kind of "repulsion" when they are not too close to each other (Lemma 3.19). After this, we will finally be able to prove Theorem 3.3. Before we follow this path, we will need an estimate involving diophantine numbers:

Proposition 3.16. *If α is diophantine, there are $m, r \in \mathbb{N}$ such that for every $j \in \mathbb{N}$*

$$|\sin(2\pi j\alpha)| > \frac{1}{m \cdot j^r}. \tag{3.12}$$

Proof. If α is diophantine, choose $n \in \mathbb{N}$ and $c > 0$ such that for every $p \in \mathbb{Z}$ and every $q \in \mathbb{N}$ we have $|q\alpha - p| > cq^{-n}$. Pick p as the integer such that $0 < |q\alpha - p| < \frac{1}{2}$. We have two possibilities:

If $|q\alpha - p| < \frac{3}{10}$, since $0 < |x| < \frac{3\pi}{5} \implies |\sin x| > \frac{|x|}{2}$ we have

$$|\sin(2\pi q\alpha)| = |\sin(2\pi(q\alpha - p))| > \pi|q\alpha - p| = \pi q\left|\alpha - \frac{p}{q}\right| > \frac{c\pi}{q^{n-1}}.$$

If $\frac{3}{10} < |q\alpha - p| < \frac{1}{2}$, by the mean value theorem, there is x_0 such that $\frac{3\pi}{5} < 2\pi|q\alpha - p| < |x_0| < \pi$ and

$$|\sin(2\pi q\alpha)| = |\sin\pi - \sin(2\pi(q\alpha - p))| = |\cos x_0|\,|\pi - 2\pi(q\alpha - p)|\,.$$

Now, x_0 lies in an interval in which $|\cos x|$ increases as $|x|$ increases, so $|\cos x_0| > \left|\cos\frac{3\pi}{5}\right|$. Furthermore, by hypothesis,

$$|\pi - 2\pi q\alpha + 2\pi p| = \pi\,|2q\alpha - 2p - 1| > \frac{c\pi}{2^n q^n}.$$

Bringing these inequalities together, we get

$$|\sin(2\pi q\alpha)| > \left|\cos\frac{3\pi}{5}\right|\frac{c\pi}{2^n q^n}.$$

Finally, if we choose a natural m such that $m > \left|\cos\frac{3\pi}{5}\right|^{-1}\frac{2^n}{c\pi} > \frac{1}{c\pi}$ and define $r = n$, we conclude that $|\sin(2\pi j\alpha)| > \frac{1}{m\cdot j^r}$ for every $j \in \mathbb{N}$.

□

Lemma 3.17. *Let α be diophantine and θ be non-resonant. There are $k_3 = k_3(m_1, \theta, \alpha)$ and $c > 0$ such that*

$$|\cos(2\pi\theta_j) - \cos(2\pi\theta_l)| \geq \frac{c}{(d+k/2)^r}\exp\left(-(k+2d)^{(2r)^{-1}}\right) \tag{3.13}$$

for all $k > k_3$ and $0 \leq j < l \leq k$

Proof. Define $i_j \in \mathbb{Z}$ by

$$i_j = \begin{cases} j, & \text{if } 0 \leq j \leq \lfloor\frac{k+1}{2}\rfloor - 1 \\ j + d - \lfloor\frac{k+1}{2}\rfloor, & \text{if } \lfloor\frac{k+1}{2}\rfloor \leq j \leq k \end{cases}$$

so that $\theta_j = \theta + (n_1 + i_j)\,\alpha$.

We then get

$$\begin{aligned} |\cos(2\pi\theta_j) - \cos(2\pi\theta_j)| &= 2\left|\sin\left(2\pi\frac{\theta_j+\theta_l}{2}\right)\sin\left(2\pi\frac{\theta_j-\theta_l}{2}\right)\right| \\ &= 2\left|\sin\left(2\pi\left(\theta + n_1\alpha + \frac{i_j+i_l}{2}\alpha\right)\right)\sin\left(2\pi(i_j-i_l)\frac{\alpha}{2}\right)\right|. \end{aligned}$$

If α is diophantine, so is $\alpha/2$ with same parameter r, so there exists $c > 0$ such that

$$\left|\sin\left(2\pi(i_j-i_l)\frac{\alpha}{2}\right)\right| > \frac{c}{|i_j-i_l|^r} \geq \frac{c}{\left(k+d-\left\lfloor\frac{k+1}{2}\right\rfloor\right)^r} \geq \frac{c}{(d+k/2)^r}.$$

Since θ is non-resonant, so is $\theta + n_1\alpha$, so there is $k_3(m_1, \alpha, \theta)$ such that for $k > k_3$

$$\begin{aligned} &\left|\sin\left(2\pi\left(\theta + n_1\alpha + \frac{i_j+i_l}{2}\alpha\right)\right)\right| \geq \exp\left(-(i_j+i_l)^{(2r)^{-1}}\right) \\ &\geq \exp\left(-\left(2k-1+2d-2\left\lfloor\frac{k+1}{2}\right\rfloor\right)^{(2r)^{-1}}\right) \geq \exp\left(-(k+2d)^{(2r)^{-1}}\right). \end{aligned}$$

Putting these two inequalities together, we complete the proof.

□

Lemma 3.18. *If α is diophantine, θ is non-resonant and $d < k^\tau$ for some $\tau < 2$, then for every $\varepsilon > 0$ there exists $k_4 = k_4(\varepsilon, \alpha, \tau, \theta, m_1)$ such that for every $k > k_4$ and every $z \in [-1, 1]$ we have*

$$\left| \frac{\prod\limits_{\substack{l=0,\dots,k \\ l\neq j}} z - \cos(2\pi\theta_l)}{\prod\limits_{\substack{l=0,\dots,k \\ l\neq j}} \cos(2\pi\theta_j) - \cos(2\pi\theta_l)} \right| \leq e^{k\varepsilon}. \tag{3.14}$$

Proof. Let $\phi(n) = 2Cn^{1-r^{-1}} \ln n$. By lemma 3.15, the points $\theta_0, \dots, \theta_k$ are $\phi-$uniformly distributed on $[0, 1]$. Applying lemma 3.14 with $f(x) = \cos(2\pi x)$, $\psi(n) = 1$ and using that for all $z \in [-1, 1]$

$$\int_0^1 \ln|z - \cos(2\pi x)|dx = -\ln 2,$$

we get

$$I_1 = \left| \prod_{\substack{l=0,\dots,k \\ l\neq j}} z - \cos(2\pi\theta_l) \right| = \exp\left(\sum_{\substack{l=0,\dots,k \\ l\neq j}} \ln|z - \cos(2\pi\theta_l)| \right)$$
$$\leq \exp\left((k+1)\left(\int_0^1 \ln|z - \cos(2\pi x)|dx + \frac{\varepsilon}{3} \right) \right) = \exp\left((k+1)\left(-\ln 2 + \frac{\varepsilon}{6} \right) \right),$$

and that

$$I_2 = \left| \prod_{\substack{l=0,\dots,k \\ l\neq j}} \cos(2\pi\theta_j) - \cos(2\pi\theta_l) \right| = \exp\left(\sum_{\substack{l=0,\dots,k \\ l\neq j}} \ln|\cos(2\pi\theta_j) - \cos(2\pi\theta_l)| \right)$$
$$\geq \exp\left((k+1)\left(-\ln 2 - \frac{\varepsilon}{3} \right) + 2\phi(k+1)\left(-(k+2d)^{(2r)^{-1}} + \ln\left(\frac{c}{(d+k/2)^r} \right) \right) \right).$$

Let $\chi(n) = \phi(n)\left(-(n+2d)^{(2r)^{-1}} + \ln\left(\frac{c}{(d+n/2)^r} \right) \right)$. Since $r \geq 1$ and $d \leq k^\tau$ for some $\tau < 2$, we have $\lim\limits_{n\to\infty} \dfrac{\chi(n)}{n} = 0$. Then for sufficiently large k

$$I_2 \geq \exp\left((k+1)\left(-\ln 2 - \frac{\varepsilon}{3} \right) \right).$$

This allows us to conclude

$$\frac{I_1}{I_2} \leq e^{(k+1)\varepsilon/2} \leq e^{k\varepsilon}$$

□

Lemma 3.19. *Assume α is diophantine and θ is non-resonant. Then for every $m_1 \in \mathbb{Z}$, $\varepsilon > 0$ and $\tau < 2$, there is $k_5(E, \alpha, \theta, m_1, \varepsilon, \tau)$ such that if $k \in \mathcal{K}$, $k > k_5$, if m_1, m_2 are both $(\gamma(E) - \varepsilon, k)$ singular and $d = |m_2 - m_1| > \frac{k+1}{2}$, then $d > k^\tau$.*

Proof. Since $k \in \mathcal{K}$, we can pick θ_0 such that $|P_k(\theta_0)| \geq \frac{1}{2}e^{k\gamma(E)}$. Setting $z_0 = \cos\theta_0$ and using equation (3.8), we get

$$\frac{1}{2}e^{k\gamma} \leq |P_k(z_0)| \leq \leq \sum_{j=0}^{k} \left| P_k(\theta_j) \frac{\prod\limits_{\substack{l=0,\dots,k \\ l\neq j}} z - \cos(2\pi\theta_l)}{\prod\limits_{\substack{l=0,\dots,k \\ l\neq j}} \cos(2\pi\theta_j) - \cos(2\pi\theta_l)} \right|.$$

Suppose, for the sake of contradiction, that $d < k^\tau$ for some $\tau < 2$. By lemmas 3.10 and 3.18, if we take $k = \max\{8k_2(E, \alpha, \varepsilon/6) + 6, k_4(\varepsilon/16, \alpha, \tau, \theta, m_1)\}$ we get

$$\frac{1}{2}e^{k\gamma} \leq |P_k(z_0)| \leq \leq \sum_{j=0}^{k} e^{k(\gamma(E)-\varepsilon/12)}e^{k\varepsilon/16} = (k+1)e^{k\gamma(E)}e^{-k\varepsilon/48}.$$

But $(k+1)e^{k\gamma(E)}e^{-k\varepsilon/48} \to 0$, so there is $K \in \mathbb{N}$ such that $k > K \implies (k+1)e^{k\gamma(E)}e^{-k\varepsilon/16} < \frac{1}{2}$. So with $k_5 = \max\{8k_2(E, \alpha, \varepsilon/6) + 6, k_4(\varepsilon/16, \alpha, \tau, \theta, m_1), K\}$ we get a contradiction, which proves the lemma. □

Using a result by Sch'nol, we can now finally prove the main result of this chapter.

Theorem 3.20 (Sch'nol)**.** *Let $\delta > \frac{1}{2}$. We say $E \in \mathbb{R}$ is a δ-generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ if there is a non zero generalized solution of $H_{\lambda,\alpha,\theta}u = Eu$ such that $|u(n)| \leq B(1 + |n|)^\delta$ for every $n \in \mathbb{Z}$ and some $B > 0$. Let A_δ be the set of δ-generalized eigenvalues of $H_{\lambda,\alpha,\theta}$ and μ a spectral measure for $H_{\lambda,\alpha,\theta}$. Then*

1. *$A_\delta \subset \sigma(H_{\lambda,\alpha,\theta})$;*
2. *$\mu(\sigma(H_{\lambda,\alpha,\theta}) \setminus A_\delta) = 0$;*
3. *$\overline{A_\delta} = \sigma(H_{\lambda,\alpha,\theta})$.*

Proof of Theorem3.3. Let $\lambda > 1$, α diophantine and θ non-resonant and let $E \in \mathbb{R}$ be such that $H_{\lambda,\alpha,\theta}u = Eu$ has a non-trivial solution with $|u(n)| \leq B(1 + |n|)$ for all $n \in \mathbb{Z}$ and some $B > 0$. We will call such u a generalized engeinfunction. By Theorem 3.20, if we prove that the generalized eigenfunctions of the $H_{\lambda,\alpha,\theta}$ are indeed eigenfunctions, that is, they do belong to ℓ^2, we get that a dense subset of the spectrum is made solely of eigenvalues. Therefore, the supercritical almost Mathieu operator has only pure point spectrum for diophantine frequencies and non-resonant phases.

Let E be a $1-$generalized eigenvalue and u the correspondent generalized eigenfunction. Then pick $n_0 \in \mathbb{Z}$ such that $n_0 \geq 0$ and $u(n_0) \neq 0$ (we can take n_0 to be 0 or 1 since a non zero solution cannot have two consecutive zero). Let $|n - n_0| > \max\{k_1(E, \gamma(E)/2, \theta, n_0), k_5(E, \alpha, \theta, n_0, \gamma(E)/2, 3/2)\} + 1$. Then for $k \geq |n - n_0| - 1$, we have that n_0 is $(\gamma(E)/2, k)-$singular.

Choose $k \in \{|n-n_0|-1, |n-n_0|, |n-n_0|+1\} \cap \mathcal{K} \neq \varnothing$. Since $|n-n_0| \geq k-1 > \dfrac{k+1}{2}$ and $|n - n_0| \leq k + 1 < k^{3/2}$, we have that n is $(\gamma(E)/2, k)-$regular. Thus we can choose an interval $[n_1, n_2]$ such that $n \in [n_1, n_2]$, $|n - n_i| \geq \frac{k}{40}$ and $|G_{[n_1,n_2]}(n, n_i)| < e^{-\gamma(E)|n-n_i|/2}$. This leads us to

$$
\begin{aligned}
&|u(n)| \leq |G_{[n_1,n_2]}(n, n_1)||u(n_1 - 1)| + |G_{[n_1,n_2]}(n, n_2)||u(n_2 + 1)| \\
<&B\left(|n_1 - 1|e^{-\gamma(E)|n-n_1|/2} + |n_2 + 1|e^{-\gamma(E)|n-n_2|/2}\right) \leq B\left(|n_1 - 1| + |n_2 + 1|\right) e^{-\gamma(E)k/80} \\
&\leq B\left(|n_1 - n| + |n - 1| + |n_2 - n| + |n + 1|\right) e^{-\gamma(E)(|n-n_0|-1)/80} \\
&\leq B\left(\frac{39k}{20} + 2|n| + 2\right) e^{-\gamma(E)(|n-n_0|-1)/80} \\
&\leq B\left(\frac{39(|n - n_0| + 1)}{20} + 2|n| + 2\right) e^{-\gamma(E)(|n-n_0|-1)/80}.
\end{aligned}
$$

Since for $\lambda > 1$ we have $\gamma(E) > 0$, the inequality we got gives us $u \in \ell^2$ and the proof is complete. □

3.3 Absence of point spectrum for special frequencies and phases

The lemma 3.17, which was essential to prove theorem 3.3, strongly used the arithmetical properties of diophantine frequencies and non-resonant phases. It is natural to wonder whether we could somehow relax these conditions and still get only pure point spectrum, or even if we could get this result for every frequency and every phase. Although we do not try to answer the former, here we will show that the answer to the latter is negative: For suitable frequencies and phases there can never be point spectrum, and even though these sets have measure zero by theorem 3.3, they are topologically "big": they form a dense G_δ.

3.3.1 Strongly liouvillian frequencies

In this subsection we show that for a dense G_δ set of frequencies the almost Mathieu operator cannot have point spectrum.

Definition 3.21. *We will say that an irrational α is* ***strongly liouvillian*** *if for every $n \in \mathbb{N}$ there exists $p, q \in \mathbb{Z}$ with $q > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{n^q}. \tag{3.15}$$

We will denote this set by $\mathbb{L}_s$.

Proposition 3.22. *$\mathbb{L}_s$ is dense G_δ.*

Proof. For each $n \in \mathbb{N}$ let $U_n = \bigcup_{q=1}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{n^q}, \frac{p}{q} + \frac{1}{n^q}\right)$. Clearly each U_n is a dense open set. Let $\{r_n\}$ be an enumeration of $\mathbb{Q}$ and $V_n = U_n \setminus \{r_n\}$, so that V_n is still a dense open set. By the definition of strongly liouvillian, $\mathbb{L}_s = \bigcap_{n=1}^{\infty} V_n$, so it is a dense G_δ. □

Lemma 3.23. *Assume that $V : \mathbb{Z} \to \mathbb{R}$ satisfies $V(n+m) = V(n)$ for some $m > 0$ and for every $n \in [-m, m-1]$. If $E \in \mathbb{R}$ and u is a solution for $u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$ we have*

$$\max\left\{\left\|\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}\right\|, \left\|\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}\right\|, \left\|\begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix}\right\|\right\} \geq \frac{1}{2}\left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\|. \tag{3.16}$$

Proof. Let A and T be, respectively, the step matrix and the transfer matrix for the eigenvalue equation $u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$. By assumption, $V(n+m) = V(n)$ for $n \in [-m, m-1]$, so $A(n+m) = A(n)$ for $n \in [-m, m-1]$. This gives us

$$T(2m) = A(2m-1) \cdots A(m)A(m-1) \cdots A(0) = (A(m-1) \cdots A(0))^2 = T(m)^2.$$

and

$$T(m)T(-m)^{-1} = A(m-1) \cdots A(0)A(-1) \cdots A(m-1) = (A(m-1) \cdots A(0))^2 = T(m)^2.$$

But $T(m) \in SL_2(\mathbb{R})$, so $T(m)^2 = \mathrm{Tr}(T(m))T(m) - I$.

If $|\mathrm{Tr}(T(m))| \leq 1$, then

$$\begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix} = T(2m)\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = T(m)^2\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$
$$= (\mathrm{Tr}(T(m))T(m) - I)\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \mathrm{Tr}(T(m))\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix} - \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$
$$\implies \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix} - \mathrm{Tr}(T(m))\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}$$
$$\implies \left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\| \leq |\mathrm{Tr}(T(m))|\left\|\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}\right\| + \left\|\begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix}\right\|$$
$$\leq \left\|\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}\right\| + \left\|\begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix}\right\|.$$

In the case $|\mathrm{Tr}(T(m))| > 1$, we use

$$\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix} = T(m)\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = T(m)T(-m)^{-1}\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix} = T(m)^2\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}$$
$$= (\mathrm{Tr}(T(m))T(m) - I)\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix} = (\mathrm{Tr}(T(m))T(-m)^{-1} - I)\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}$$
$$= \mathrm{Tr}(T(m))\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} - \begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}$$
$$\implies \mathrm{Tr}(T(m))\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix} + \begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}$$
$$\implies \left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\| < |\mathrm{Tr}(T(m))|\left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}\right\| + \left\|\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}\right\|.$$

So at least one of the three following inequalities must hold: $\left\|\begin{pmatrix} u(-m) \\ u(-m-1) \end{pmatrix}\right\| \geq \frac{1}{2}\left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\|$, $\left\|\begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}\right\| \geq \frac{1}{2}\left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\|$ or $\left\|\begin{pmatrix} u(2m) \\ u(2m-1) \end{pmatrix}\right\| \geq \frac{1}{2}\left\|\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}\right\|$. □

We now state a result of Gordon that was originaly made for the continuous case, but adapted to the discrete case by Avron and Simon [8, 32]:

Lemma 3.24. *Assume $V : \mathbb{Z} \to \mathbb{R}$ is bounded and that there is a sequence (q_k) of positive integers such that $\lim_{k\to\infty} q_k = \infty$ and for every $C > 0$*

$$\lim_{k\to\infty} \max_{0\leq n\leq q_k-1} |V(n) - V(n \pm q_k)|C^{q_k} = 0.$$

Then for every $E \in \mathbb{R}$ the equation $u(n+1)+u(n-1)+V(n)u(n) = Eu(n)$ has no square-summable solution.

Proof. Consider a solution u for $u(n+1)+u(n-1)+V(n)u(n) = Eu(n)$ and a solution u_k for the equation $u_k(n+1)+u_k(n-1)+V_k(n)u(n) = Eu_k(n)$ with $u_k(-1) = u(-1)$ and $u_k(0) = u(0)$, where V_k is the q_k-periodic function such that $V_k(n) = V(n)$ for $n \in [0, q_k - 1]$.

Let $T(n)$ be as in the proof of the previous lemma and $T_k(n)$ the transfer matrix associated with V_k, same for $A(n)$ and $A_k(n)$. We then have

$$\max_{-q_k \le n \le 2q_k-1} \left\| \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} - \begin{pmatrix} u_k(n) \\ u_k(n-1) \end{pmatrix} \right\| \le \max_{-q_k \le n \le 2q_k-1} \|T(n) - T_k(n)\| \left\| \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \right\|$$

Let $C = \sqrt{(|E| + \|V\|_\infty)^2 + 2}$. Then

$$\max_{-q_k \le n \le 2q_k-1} \|T(n) - T_k(n)\| \le (2q_k - 1)C^{2q_k-2} \max_{-q_k \le n \le 2q_k-1} |V(n) - V_k(n)|$$

But $\max_{-q_k \le n \le 2q_k-1} |V(n) - V_k(n)| = \max_{0 \le n \le q_k-1} |V(n \pm q_k) - V(n)|$, since V_k is q_k-periodic and coincides with V for $n \in [0, q_k - 1]$. So we have

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \max_{-q_k \le n \le 2q_k-1} \left\| \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} - \begin{pmatrix} u_{(k)}(n) \\ u_{(k)}(n-1) \end{pmatrix} \right\| \\ &\le \lim_{k\to\infty} (2q_k - 1)C^{2q_k-2} \max_{0 \le n \le q_k-1} |V(n \pm q_k) - V(n)| \\ &= \lim_{k\to\infty} \frac{2q_k - 1}{2^{2q_k-2}} (2C)^{2q_k-2} \max_{0 \le n \le q_k-1} |V(n \pm q_k) - V(n)| = 0. \end{aligned}$$

By lemma 3.23,

$$\max \left\{ \left\| \begin{pmatrix} u_k(-q_k) \\ u_k(-q_k - 1) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_k(q_k) \\ u_k(q_k - 1) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_k(2q_k) \\ u_k(2q_k - 1) \end{pmatrix} \right\| \right\} \ge \frac{1}{2} \left\| \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \right\|.$$

So u_k does not decay and therefore neither does u, which rules out the possibility of u being square-summable.

□

Theorem 3.25. *If $\alpha \in \mathbb{L}_s$, then $H_{\lambda,\alpha,\theta}$ has no point spectrum.*

Proof. Let p_n/q_n be the continued fraction approximants of α, so that $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{n^{q_n}}$. So for any $\theta \in \mathbb{R}$ and any $C > 0$ we have

$$\max_{0\le m\le q_n-1} |2\lambda\cos(2\pi(\theta+m\alpha)-2\lambda\cos(2\pi(\theta+(m\pm q_n)\alpha)|C^{q_n}$$
$$=\max_{0\le m\le q_n-1} 4\lambda|\sin(\pi(2\theta+(2m\pm q_n))\alpha)\sin(\mp\pi q_n\alpha)|C^{q_n}\le 4\pi\lambda\|q_n\alpha\|_{\mathbb{T}}C^{q_n}.$$

So, by construction

$$0\le\lim_{n\to\infty}\max_{0\le m\le q_n-1}|2\lambda\cos(2\pi(\theta+m\alpha)-2\lambda\cos(2\pi(\theta+(m\pm q_n)\alpha)|C^{q_n}$$
$$\le\lim_{n\to\infty}4\pi\lambda|p_n-q_n\alpha|C^{q_n}\le\lim_{n\to\infty}4\pi\lambda\frac{C^{q_n}}{n^{q_n-1}}=0.$$

Thus, for strongly liouvillian α, the potential of the almost Mathieu opereator satisfies the assumptions of lemma 3.24, and therefore cannot have point spectrum.

□

3.3.2 Strongly resonant phases

In this subsection, we show that for a fixed frequency, there is a dense G_δ set of frequencies for which the almost Mathieu operator cannot have point spectrum.

Definition 3.26. *Fix an irrational α and $\lambda>0$. Then for $B=4\ln(3+4\lambda)$ and $n\in\mathbb{N}$ let*

$$\Phi_n=\left\{\theta\in\mathbb{R}\;\middle|\;|\sin(2\pi(\theta+n\alpha))|<e^{-Bn}\right\}. \tag{3.17}$$

We then define the set of ***strongly resonant phases*** *by*

$$\Phi=\limsup_n\Phi_n=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\Phi_k.$$

Proposition 3.27. *Φ is a dense G_δ.*

Proof. By continuity of $\theta\mapsto|\sin(2\pi(\theta+n\alpha))|$, we have that each Φ_n is open. Thus, so is each $\bigcup_{k=n}^{\infty}\Phi_k$. Also, for each n, it is clear that $U_n=\{m-n\alpha\mid m\in\mathbb{Z}\}\subset\Phi_n$. But we know that the set $\bigcup_{k=n}^{\infty}U_k=\{m-k\alpha\mid m\in\mathbb{Z}\text{ and }k\ge n\}$ is dense for each n, so each $\bigcup_{k=n}^{\infty}\Phi_k$ is a dense open set. Hence, Φ is a dense G_δ. □

Theorem 3.28. *For $\theta\in\Phi$, the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ has no point spectrum*

Proof. First of all, let $V(n) = 2\lambda \cos 2\pi\,(\theta + n\alpha)$, then

$$|V(2m-n) - V(n)| = 4\lambda |\sin 2\pi(m-n)\alpha \sin\,(\theta + m\alpha)| \le 4\lambda |\sin\,(\theta + m\alpha)|$$

Since $\theta \in \Phi$, we can pick an increasing sequence (m_k) such that $|V(2m_k - n) - V(n)| \le 4\lambda e^{-Bm_k}$ for every k.

Suppose $H_{\lambda,\alpha,\theta}u = Eu$ has a non zero solution which is square-summable, we can then choose such eigenfunction satisfying $\|u\| = 1$. Furthermore, since E is in the spectrum, $|E| \le 2 + 2\lambda$. Let $u_k(n) = u(2m_k - n)$ and the $W[u,v](n) = u(n+1)v(n) - u(n)v(n+1)$. We have

$$\begin{aligned}
&|W[u,u_k](n) - W[u,u_k](n-1)| \\
&= |u(n+1)u_k(n) - u(n)u_k(n+1) - u(n)u_k(n-1) + u(n-1)u_k(n)| \\
&= |u_k(n)(u(n+1) + u(n-1)) - u(n)(u_k(n+1) + u_k(n-1))| \\
&= |u_k(n)u(n)||V(n) - V(2m_k - n)| \le 4\lambda e^{-Bm_k}.
\end{aligned} \tag{3.18}$$

Since u and u_k are in ℓ^2, so does $W[u,u_k]$ and, by Cauchy–Bunyakovsky–Schwarz inequality, $\|W[u,u_k]\| \le 2$ since both u and u_k have unit norm. So for some j such that $|j| \le e^{Bm_k/2}$ we must have $|W[u,u_k](j)| \le e^{-Bm_k/2}$. But by (3.18), using triangle inequality we get that for $|n| \le e^{Bm_k/2}$

$$|W[u,u_k](n)| \le 2e^{Bm_k/2} \cdot 4\lambda e^{-Bm_k} + e^{-Bm_k/2} = (8\lambda + 1)e^{-Bm_k/2}$$

In particular, this holds for $n = m_k$, since $m_k \le e^{Bm_k/2}$ for k large enough.

Now define $u_k^{\pm} = u \pm u_k$. Clearly, $W[\cdot,\cdot]$ is bilinear and anticommutative. This gives us $W[u_k^-, u_k^+] = W[u,u] - W[u_k,u] + W[u,u_k] - W[u_k,u_k] = 2W[u,u_k]$. Since $u_k^-(m_k) = 0$, we get

$$|u_k^+(m_k)u_k^-(m_k+1)| = |W[u_k^-, u_k^+](m_k)| \le 2(8\lambda+1)e^{-Bm_k/2}$$

So either $|u_k^+(m_k)| \le \sqrt{2(8\lambda+1)}e^{-Bm_k/4}$ or $|u_k^-(m_k+1)| \le \sqrt{2(8\lambda+1)}e^{-Bm_k/4}$

Now define $\Psi_k^{\pm}(n) = \begin{pmatrix} u_k^{\pm}(n+1) \\ u_k^{\pm}(n) \end{pmatrix}$. Consider the case where $|u_k^+(m_k)| \le \sqrt{2(8\lambda+1)}e^{-Bm_k/4}$ holds, then $\|\Psi_k^+(m_k)\| = \sqrt{|u_k^+(m_k)|^2 + |u_k^+(m_k+1)|^2}$. But

$$
\begin{aligned}
0 &= u(m_k+1) + u(m_k-1) + (V(m_k)-E)u(m_k) \\
&= u(m_k+1) + u(2m_k-(m_k+1)) + \frac{1}{2}(V(m_k)-E)(u(m_k+1)+u(2m_k-m_k)) \\
&= u_k^+(m_k+1) + \frac{1}{2}(V(m_k)-E)u_k^+(m_k).
\end{aligned}
$$

so $\|\Psi_k^+(m_k)\| = |u_k^+(m_k)|\sqrt{1+(V(m_k)-E)^2/4} \leq \sqrt{2(8\lambda+1)}e^{-Bm_k/4}\sqrt{1+(1+2\lambda)^2}$. If $|u_k^-(m_k+1)| \leq \sqrt{2(8\lambda+1)}e^{-Bm_k/4}$, then $\|\Psi_k^-(m_k)\| = |u_k^-(m_k+1)| \leq \sqrt{2(8\lambda+1)}e^{-Bm_k/4}$, since $u_k^-(m_k)=0$. Either way, for at least one sign, there is $C>0$ such that

$$\|\Psi_k^{\pm}(m_k)\| \leq Ce^{-Bm_k/4}$$

Let $\Psi(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$ and $\Psi_k(n) = \begin{pmatrix} u_k(n+1) \\ u_k(n) \end{pmatrix}$, so $\Psi_k^{\pm}(n) = \Psi(n) \pm \Psi_k(n)$. Also let

$$T_k^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & E-V(1) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & E-V(m_k) \end{pmatrix}$$

and

$$T_k^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & E-V(2m_k-(m_k-1)) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & E-V(2m_k) \end{pmatrix}$$

So $T_k^{(1)}\Psi(m_k) = \Psi(0)$ and $T_k^{(2)}\Psi_k(m_k) = \Psi_k(0)$. By the same reasoning used in the demonstration of lemma 3.24, we have that

$$\|T_k^{(1)} - T_k^{(2)}\| \leq m_k \left(\max_{n\in\mathbb{Z}} \left\| \begin{pmatrix} 0 & 1 \\ -1 & E-V(n) \end{pmatrix} \right\| \right)^{m_k-1} \left(\max_{n\in\mathbb{Z}} \left\| \begin{pmatrix} 0 & 0 \\ 0 & V(2m_k-n)-V(n) \end{pmatrix} \right\| \right)$$

But for E in the spectrum, $|E| \leq 2\lambda$, so

$$
\begin{aligned}
&\left\| \begin{pmatrix} 0 & 1 \\ -1 & E-V(n) \end{pmatrix} \right\| = \sqrt{2+(E-V(n))^2} \leq \sqrt{16\lambda^2+16\lambda+6} \\
&< \sqrt{16\lambda^2+24\lambda+9} = 4\lambda+3 = e^{B/4},
\end{aligned}
$$

and $|V(2m_k-n)-V(n)| \leq 4\lambda e^{-Bm_k}$. So $\|T_k^{(1)} - T_k^{(2)}\| < m_k e^{B(m_k-1)/4} \cdot 4\lambda e^{-Bm_k} \leq 4\lambda m_k e^{-3Bm_k/4}$.

Finally,

$$\Psi_k^{\pm}(0) = T_k^{(1)}\Psi(m_k) \pm T_k^{(2)}\Psi_k(m_k) = T_k^{(1)}\Psi_k^{\pm}(m_k) \pm (T_k^{(2)} - T_k^{(1)})\Psi_k(m_k).$$

So

$$\|\Psi_k^{\pm}(0)\| \le \left\|T_k^{(1)}\Psi_k^{\pm}(m_k)\right\| + \left\|\left(T_k^{(2)} - T_k^{(1)}\right)\Psi_k(m_k)\right\|$$
$$\le (\sqrt{16\lambda^2+16\lambda+6})^{m_k} C e^{-Bm_k/4} + 4\lambda m_k e^{-3Bm_k/4}\|\Psi_k(m_k)\|.$$

Taking $k \to \infty$ we get

$$\|\Psi(0)\| \le \|\Psi_k(0)\| + \|\Psi_k^{\pm}(0)\| = \|\Psi(2m_k-1)\| + \|\Psi_k^{\pm}(0)\| \to 0$$

so $\|\Psi(0)\| = 0 \implies u(1) = u(0) = 0 \implies u \equiv 0$, which is a contradiction. □

4 Critical case

We have given, up to measure zero sets of frequency and phase, a classification of the spectrum of $H_{\lambda,\alpha,\theta}$ for $\lambda > 1$ (and, by duality results, of $\lambda < 1$). We will now focus on the critical case, that is, when $\lambda = 1$.

In the first section of this chapter, following Last in [30], we will use two results of [10] to derive the measure of the spectrum for a full measure set of irrational frequencies. In particular, using Theorem 2.10, we will conclude that for almost every α and almost every θ the operator $H_{1,\alpha,\theta}$ has purely singular continuous spectrum.

In the second section we will use Avila's work in [3] to show that $H_{1,\alpha,\theta}$ can never have eigenvalues, except maybe for countably many θ. That property, alongside with the zero measure of its spectrum, gives us a stronger result than the one obtained in the previous section.

4.1 The measure of the spectrum

In this section, in order to study the measure of $\sigma(H_{\lambda,\alpha,\theta})$, we will use the periodic approximants of the operator, that is, we will choose a rational approximant p/q for α. Since in this case the spectrum is not independent of θ, we define for each $\lambda > 0$ and $\alpha \in \mathbb{R}$ the sets $S_+(\lambda,\alpha) = \bigcup_{\theta\in\mathbb{T}} \sigma(H_{\lambda,\alpha,\theta})$ and $S_-(\lambda,\alpha) = \bigcap_{\theta\in\mathbb{T}} \sigma(H_{\lambda,\alpha,\theta})$. For irrational α, both sets coincide with $\sigma(H_{\lambda,\alpha,\theta})$. The main result of this section will be

Theorem 4.1. *Let α be an irrational such that there is a sequence of rationals $\frac{p_n}{q_n}$ for which*

$$\lim_{n\to\infty} q_n^2 \left|\alpha - \frac{p_n}{q_n}\right| = 0.$$

Then $\nu(\sigma(H_{\lambda,\alpha,\theta})) = 4|1-\lambda|$.

The set of such α is easily seen to be a dense G_δ. Furthermore, the theory of continued fractions (see [28]) tells us that such irrationals are precisely those with unbounded elements in its continued fraction representation, so it is a set of full measure. Therefore the complement of the set of such α's is negligible both in a topological sense and in a measure-theoretic sense.

We will now present some Floquet Theory's results that we will use to study the periodic approximants $H_{\lambda,p/q,\theta}$. Afterwards, there will be the statement of two important theorems about the sets S_+ and S_-. Throughout the text, when we talk about a rational p/q, we will always assume that p and q are coprime integers and $q > 0$.

Fixed $\lambda > 0$ and a rational p/q, define the function

$$D_{\lambda,p/q,\theta}(E) = \mathrm{Tr}(A_1(E,\theta)A_2(E,\theta)\cdots A_q(E,\theta)),$$

where $A_n(E,\theta) = \begin{pmatrix} E - 2\lambda\cos(\pi(\theta + np/q)) & -1 \\ 1 & 0 \end{pmatrix}$.

Proposition 4.2. *$D_{\lambda,p/q,\theta}(E)$ is a monic polynomial of order q in E with the properties:*

1. *$D_{\lambda,p/q,\theta}(E)$ has q real distinct zeros;*
2. *$D_{\lambda,p/q,\theta}(E) \geq 2$ at its maxima points and $D_{\lambda,p/q,\theta}(E) \leq -2$ at its minima points;*
3. *$D^{-1}_{\lambda,p/q,\theta}([-2,2]) = \sigma(H_{\lambda,p/q,\theta})$, so $\sigma(H_{\lambda,p/q,\theta})$ is the union of q bands (closed intervals) with pairwise disjoint interiors and $D_{\lambda,p/q,\theta}$ is strictly monotone in each band.*

For a proof, see, e.g., chapter 7 of [33].

Using the ciclicity of the trace and that $D_{\lambda,p/q,\theta}(E)$ is a polynomial of order q in $e^{i\theta}$ and $e^{-i\theta}$, it is easy to show that (see [11]):

$$D_{\lambda,p/q,\theta}(E) = D_{\lambda,p/q,1/q}(E) + 2\lambda^q\cos(2\pi q\theta). \tag{4.1}$$

Let $f_{\lambda,p/q}(E) = D_{\lambda,p/q,1/q}(E)$. Then by equation 4.1 and property 3 of $D_{\lambda,p/q,\theta}(E)$, we obtain that $S_+(\lambda,p/q) = f^{-1}_{\lambda,p/q}([-2-2\lambda^q, 2+2\lambda^q])$ for every λ; that $S_-(\lambda,p/q) = f^{-1}_{\lambda,p/q}([-2+2\lambda^q, 2-2\lambda^q])$ for $\lambda \leq 1$; and that $S_-(\lambda,p/q) = \varnothing$ for $\lambda > 1$. We also have that $f_{\lambda,p/q}(E) \geq 2 + 2\lambda^q$ at its maxima points and $f_{\lambda,p/q}(E) \leq -2 - 2\lambda^q$ at its minima points, so $S_+(\lambda,p/q)$ is made of q bands with pairwise disjoint interiors and $f_{\lambda,p/q}$ is strictly monotone in each band.

Since $f_{\lambda,p/q}$ is a polynomial in E, $S_+(\lambda,p/q)$ is always compact, so the set $G(\lambda,\alpha)$ defined by $G(\lambda,\alpha) = [\min S_+(\lambda,p/q), \max S_+(\lambda,p/q)] \setminus S_+(\lambda,p/q)$ is a bounded open set and can therefore be written as the union of countably many disjoint open intervals. We call each of these intervals a gap $S_+(\lambda,p/q)$.

Now we have the basic facts about the periodic almost Mathieu operators, we will enunciate the two aforementioned theorems about the sets S_+ and S_-. For the proof of Theorem 4.3, see [10].

Theorem 4.3. *For $0 \leq \lambda \leq 1$ we have*

$$\nu(S_-(\lambda,\alpha)) = 4(1-\lambda) \tag{4.2}$$

$$4(1-\lambda) \leq \nu(S_+(\lambda,\alpha)) \leq 4(1-\lambda) + 4\pi\lambda^{q/2} \tag{4.3}$$

Theorem 4.4. *For every $\lambda > 0$, there is $\delta > 0$ such that if $E \in S_+(\lambda, \alpha)$ and $|\alpha - \alpha'| < \delta$, then there is $E' \in S_+(\lambda, \alpha')$ such that $|E - E'| < 16\sqrt{\lambda|\alpha - \alpha'|}$.*

Proof. The idea of the proof is to use approximate eigenfunctions for $H_{\lambda,\alpha,\theta} - E$ and multiply them by a test function to construct a new vector in ℓ^2 and show that this new vector is not "too far" from being an approximate eigenfunction of $H_{\lambda,\alpha',\theta'} - E$ for suitable θ'. Therefore, we will come to the conclusion that there is E' in the spectrum of $H_{\lambda,\alpha',\theta'}$ that is not "too far" from E.

For $E \in S_+(\lambda, \alpha)$ and $\varepsilon > 0$, choose $u_\varepsilon \in \ell^2$ such that $\|(H_{\lambda,\alpha,\theta} - E)u_\varepsilon\| < \varepsilon\|u_\varepsilon\|$. Now for $j \in \mathbb{Z}$ and $L > 0$ we define the ℓ^2 function $\eta_{j,L}(n) = \begin{cases} 1 - \frac{|n-j|}{L}, & \text{if } |n-j| \le L \\ 0, & \text{if } |n-j| > L. \end{cases}$

Using $l = \lfloor L \rfloor$, note that

$$\sum_{j\in\mathbb{Z}} \eta_{j,L}(n)^2 = \sum_{j=n-l}^{n+l} \left(1 - \frac{|n-j|}{L}\right)^2 = \sum_{j=-l}^{l} \left(1 - \frac{|j|}{L}\right)^2 = 1 + 2\sum_{n=-l}^{l} \left(1 - \frac{j}{L}\right)^2$$
$$= 1 + 2\sum_{j=1}^{l} \left(1 - 2\frac{j}{L} + \frac{j^2}{L^2}\right) = 1 + 2l - \frac{4}{L} \cdot \frac{l(l+1)}{2} + \frac{2}{L^2} \cdot \frac{l(l+1)(2l+1)}{6}$$
$$= 1 + \frac{l(6L^2 - 6Ll + 2l^2 - 6L + 3l + 1)}{3L^2}$$

So $M_L(n) = \sum_{j\in\mathbb{Z}} \eta_{j,L}(n)^2$ is independent of n. Moreover, since $\lim_{L\to\infty} \frac{l}{L} = 1$, we have $\lim_{L\to\infty} \frac{M_L}{l} = \frac{2}{3}$. Now

$$\sum_{j\in\mathbb{Z}} \|\eta_{j,L}(H_{\lambda,\alpha,\theta} - E)u_\varepsilon\|^2 = \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \eta_{j,L}(n)^2 [(H_{\lambda,\alpha,\theta} - E)u_\varepsilon](n)^2$$
$$= \sum_{n\in\mathbb{Z}} \left([(H_{\lambda,\alpha,\theta} - E)u_\varepsilon](n)^2 \sum_{j\in\mathbb{Z}} \eta_{j,L}(n)^2 \right) = \sum_{n\in\mathbb{Z}} M_L [(H_{\lambda,\alpha,\theta} - E)u_\varepsilon](n)^2$$
$$= M_L \|(H_{\lambda,\alpha,\theta} - E)u_\varepsilon\|^2 \le M_L \varepsilon^2 \|u_\varepsilon\|^2.$$

We also have, since M_L is independent of n,

$$M_L\|u_\varepsilon\|^2 = \left(\sum_{j\in\mathbb{Z}} \eta_{j,L}(n)^2\right)\left(\sum_{n\in\mathbb{Z}} u_\varepsilon(n)^2\right) = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \eta_{j,L}(n)^2 u_\varepsilon(n)^2 = \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2$$
$$\implies \sum_{j\in\mathbb{Z}} \|\eta_{j,L}(H_{\lambda,\alpha,\theta} - E)u_\varepsilon\|^2 \le \varepsilon^2 \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2. \tag{4.4}$$

Let's now look at the expression $[(H_{1,\alpha,\theta} - E)(\eta_{j,L}u_\varepsilon)](n)$. We have

$$
\begin{aligned}
&[(H_{1,\alpha,\theta} - E)(\eta_{j,L} u_\varepsilon)](n) \\
&= \eta_{j,L}(n+1)u_\varepsilon(n+1) + \eta_{j,L}(n-1)u_\varepsilon(n-1) + (2\cos(2\pi(n\alpha+\theta) - E)\eta_{j,L}(n)u_\varepsilon(n) \\
&= \eta_{j,L}(n)\left(u_\varepsilon(n+1) + u_\varepsilon(n-1) + (2\cos(2\pi(n\alpha+\theta) - E)u_\varepsilon(n)\right) \\
&\qquad + (\eta_{j,L}(n+1) - \eta_{j,L}(n))u_\varepsilon(n+1) + (\eta_{j,L}(n-1) - \eta_{j,L}(n))u_\varepsilon(n-1) \\
&= \eta_{j,L}(n)[(H_{\lambda,\alpha,\theta} - E)u_\varepsilon](n) + (\eta_{j,L}(n+1) - \eta_{j,L}(n))u_\varepsilon(n+1) \\
&\qquad\qquad + (\eta_{j,L}(n-1) - \eta_{j,L}(n))u_\varepsilon(n-1).
\end{aligned}
$$

Now, let's look at the three possible cases for the value of $\eta_{j,L}(n+1) - \eta_{j,L}(n)$. If $j - l \leq n < j + l$, then

$$
\eta_{j,L}(n+1) - \eta_{j,L}(n) = \left(1 - \frac{|n+1-j|}{L}\right) - \left(1 - \frac{|n-j|}{L}\right) = \frac{|n+1-j| - |n-j|}{L} = \pm\frac{1}{L}.
$$

If $n = j + l$ or $n = j - l - 1$, then $\eta_{j,L}(n+1)$ or $\eta_{j,L}(n)$ is equal to zero, respectively, so

$$
\eta_{j,L}(n+1) - \eta_{j,L}(n) = \pm\left(1 - \frac{l}{L}\right) = \pm\frac{L-l}{L}.
$$

Finally, if $n > j + l$ or $n < j - l - 1$, both integers have image zero, whence $\eta_{j,L}(n+1) - \eta_{j,L}(n) = 0$. An analogous analysis can be made for $\eta_{j,L}(n-1) - \eta_{j,L}(n)$. Note that in the two first cases we have $|\eta_{j,L}(n+1) - \eta_{j,L}(n)| \leq \dfrac{1}{L}$.

Define the operator $T_{j,L}$ by

$$
[T_{j,L}u](n) = (\eta_{j,L}(n+1) - \eta_{j,L}(n))u(n+1) + (\eta_{j,L}(n-1) - \eta_{j,L}(n))u(n-1).
$$

We now compute

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \|T_{j,L}u_\varepsilon\|^2 = \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |(\eta_{j,L}(n+1) - \eta_{j,L}(n))u_\varepsilon(n+1) + (\eta_{j,L}(n-1) - \eta_{j,L}(n))u_\varepsilon(n-1)|^2 \\
&\leq 2\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |(\eta_{j,L}(n+1) - \eta_{j,L}(n))u_\varepsilon(n+1)|^2 + 2\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |(\eta_{j,L}(n-1) - \eta_{j,L}(n))u_\varepsilon(n-1)|^2 \\
&\leq 2\sum_{j\in\mathbb{Z}}\sum_{n=j-l-1}^{j+l} \frac{1}{L^2}|u_\varepsilon(n+1)|^2 + 2\sum_{j\in\mathbb{Z}}\sum_{n=j-l}^{j+l+1} \frac{1}{L^2}|u_\varepsilon(n-1)|^2 \\
&= 2\sum_{n\in\mathbb{Z}}\sum_{j=n-l}^{j+l+1} \frac{1}{L^2}|u_\varepsilon(n+1)|^2 + 2\sum_{n\in\mathbb{Z}}\sum_{n=n-l-1}^{n+l} \frac{1}{L^2}|u_\varepsilon(n-1)|^2 \\
&= \frac{4(l+1)}{L^2}\sum_{n\in\mathbb{Z}} |u_\varepsilon(n+1)|^2 + |u_\varepsilon(n-1)|^2 = \frac{8(l+1)}{L^2}\|u_\varepsilon\|^2. \qquad (4.5)
\end{aligned}
$$

Since $(H_{\lambda,\alpha,\theta} - E)\eta_{j,L} = \eta_{j,L}(H_{\lambda,\alpha,\theta} - E) + T_{j,L}$, using (4.4), (4.5) and $\|u+v\|^2 \leq 2\|u\|^2 + 2\|v\|^2$ we get

$$\sum_{j\in\mathbb{Z}} \|(H_{1,\alpha,\theta} - E)\eta_{j,L}u_\varepsilon\|^2 \leq 2\sum_{j\in\mathbb{Z}} \|\eta_{j,L}(H_{\lambda,\alpha,\theta} - E)u_\varepsilon\|^2 + 2\sum_{j\in\mathbb{Z}} \|T_{j,L}u_\varepsilon\|^2$$
$$\leq 2\varepsilon^2 \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2 + \frac{16(l+1)}{L^2}\|u_\varepsilon\|^2 = 2\varepsilon^2 \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2 + \frac{16(l+1)}{L^2 M_L} \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2.$$

Since $\lim\limits_{L\to\infty} \dfrac{16(l+1)}{M_L} = 24$ there is L_0 such that $\dfrac{16(l+1)}{L^2 M_L} < 25L^{-2}$ whenever $L > L_0$. For such L's we have

$$\sum_{j\in\mathbb{Z}} \|(H_{1,\alpha,\theta} - E)\eta_{j,L}u_\varepsilon\|^2 < (2\varepsilon^2 + 25L^{-2}) \sum_{j\in\mathbb{Z}} \|\eta_{j,L}u_\varepsilon\|^2.$$

So for at least one $k \in \mathbb{Z}$

$$\|(H_{1,\alpha,\theta} - E)\eta_{k,L}u_\varepsilon\|^2 < (2\varepsilon^2 + 25L^{-2})\|\eta_{k,L}u_\varepsilon\|^2$$
$$\implies \|(H_{1,\alpha,\theta} - E)\eta_{k,L}u_\varepsilon\| < \|\eta_{k,L}u_\varepsilon\|\sqrt{(2\varepsilon^2 + 25L^{-2})} \leq (\varepsilon\sqrt{2} + 5L^{-1})\|\eta_{k,L}u_\varepsilon\|$$

Now choose θ' such that $k\alpha + \theta = k\alpha' + \theta'$. If $|n-k| \leq L$, then

$$2\lambda|\cos(2\pi(n\alpha+\theta)) - \cos(2\pi(n\alpha'+\theta'))|$$
$$= 4\lambda|\sin(\pi(n(\alpha+\alpha') + \theta + \theta'))\sin(\pi(n(\alpha-\alpha') + \theta - \theta'))|$$
$$\leq 4\lambda|\sin(\pi(n(\alpha-\alpha') + \theta - \theta'))| = 4\lambda|\sin(\pi(n-k)(\alpha-\alpha'))|$$
$$\leq 4\lambda\pi|n-k||\alpha-\alpha'| \leq 4\lambda\pi L|\alpha-\alpha'|.$$

We then get

$$\|(H_{\lambda,\alpha',\theta'} - E)\eta_{k,L}u_\varepsilon\| \leq \|(H_{\lambda,\alpha',\theta'} - H_{\lambda,\alpha,\theta})\eta_{k,L}u_\varepsilon\| + \|(H_{\lambda,\alpha,\theta} - E)\eta_{k,L}u_\varepsilon\|$$

$$< (4\lambda\pi L|\alpha-\alpha'| + \varepsilon\sqrt{2} + 5L^{-1})\|\eta_{k,L}u_\varepsilon\|.$$

Now let $\delta = (\lambda\pi L_0^2)^{-1} < \dfrac{5}{4\lambda\pi L_0^2}$, so that if $|\alpha-\alpha'| < \delta$ then $\sqrt{\dfrac{5}{4\lambda\pi|\alpha-\alpha'|}} > L_0$. So, for $|\alpha-\alpha'| < \delta$, we can choose $L = \sqrt{\dfrac{5}{4\lambda\pi|\alpha-\alpha'|}}$ and finally get

$$\|(H_{\lambda,\alpha',\theta'} - E)\eta_{k,L}u_\varepsilon\| < (\varepsilon\sqrt{2} + 4\sqrt{5\lambda\pi|\alpha - \alpha'|}).$$

Since $4\sqrt{5\pi} < 16$ and ε is arbitrary, if $|\alpha - \alpha'| < \delta = (\lambda\pi L_0^2)^{-1}$ then there is $E' \in \sigma(H_{\lambda,\alpha',\theta'})$ such that, $|E - E'| < 16\sqrt{\lambda|\alpha - \alpha'|}$ and the proof is complete.

□

Corollary 4.4.1. *For every $\lambda > 0$, there is $\delta > 0$ such that if $|\alpha-\alpha'| < \delta$ then the inequalities $|\max S_+(\lambda,\alpha) - \max S_+(\lambda,\alpha')| < 16\sqrt{\lambda|\alpha - \alpha'|}$ and $|\min S_+(\lambda,\alpha) - \min S_+(\lambda,\alpha')| < 16\sqrt{\lambda|\alpha - \alpha'|}$ hold.*

Proof. Let $M = \max S_+(\lambda,\alpha)$. So by Theorem 4.4 $S_+(\lambda,\alpha')$ can only have elements E' not greater then $M + 16\sqrt{\lambda|\alpha - \alpha'|}$ (otherwise there would be $E \in S_+(\lambda,\alpha)$ such that $E > M$), so we get $\max S_+(\lambda,\alpha') < M + 16\sqrt{2\lambda|\alpha - \alpha'|}$. Again by Theorem 4.4, we know that there is $M' \in S_+(\lambda,\alpha')$ such that $M - 16\sqrt{\lambda|\alpha - \alpha'|} < M' < M + 16\sqrt{\lambda|\alpha - \alpha'|}$, so $\max S_+(\lambda,\alpha') > M - 16\sqrt{\lambda|\alpha - \alpha'|}$. Thus we have $|\max S_+(\lambda,\alpha) - \max S_+(\lambda,\alpha')| < 16\sqrt{\lambda|\alpha - \alpha'|}$.

The proof for the minimum is entirely analogous. □

Corollary 4.4.2. *For every For every $\lambda > 0$, there is $\delta > 0$ such that if $|\alpha - \alpha'| < \delta$, then for every gap g of $S_+(\lambda,\alpha)$ with $\nu(g) > 32\sqrt{\lambda|\alpha - \alpha'|}$ and midpoint E_g, there is a gap g' in $S_+(\lambda,\alpha')$ such that $E_g \in g'$ and $\nu(g') > \nu(g) - 32\sqrt{2\lambda|\alpha - \alpha'|}$.*

Proof. Since $|E_g - E| > 16\sqrt{\lambda|\alpha - \alpha'|}$ for every $E \in S_+(\lambda,\alpha)$, $E_g \notin S_+(\lambda,\alpha')$. We also know, by the definition of the gaps, that $E_g - \nu(g)/2 \in S_+(\lambda,\alpha)$ and $E_g + \nu(g)/2 \in S_+(\lambda,\alpha)$. So there are elements E', E'' of $S_+(\lambda,\alpha')$ that satisfy the inequalities $|E' - (E_g - \nu(g)/2)| < 16\sqrt{\lambda|\alpha - \alpha'|}$ and $|E'' - (E_g + \nu(g)/2)| < 16\sqrt{\lambda|\alpha - \alpha'|}$, so $E' < E_g < E''$. Therefore, $E_g \in G(\lambda,\alpha')$. Then there is a gap g' of $S_+(\lambda,\alpha')$ such that $E_g \in g' \subset (E', E'')$. Now g' is of the form (a, b), with $a, b \in S_+(\lambda,\alpha')$. By Theorem 4.4, we have $a < E_g - \nu(g)/2 + 16\sqrt{\lambda|\alpha - \alpha'|}$ and $b > E_g + \nu(g)/2 - 16\sqrt{\lambda|\alpha - \alpha'|}$, whence $\nu(g') > \nu(g) - 32\sqrt{2\lambda|\alpha - \alpha'|}$. □

Now we will proceed to gather the results that will allow us to prove Theorem4.1.

Lemma 4.5. *For a rational p/q, let $E_1, \ldots, E_q$ be the zeros of $f_{1,p/q}(E)$. We have*

$$\sum_{j=1}^{q} \frac{1}{|f'_{1,p/q}(E_j)|} = \frac{1}{q}. \tag{4.6}$$

Proof. Let $\lambda < 1$ and let $[a_1, b_1], \ldots, [a_q, b_q]$ be the q bands of $f^{-1}_{\lambda,p/q}([-2+2\lambda^q, 2-2\lambda^q])$. We have $|f(b_j) - f(a_j)| = 4 - 4\lambda^q$ for each j. By the Mean Value Theorem there is $E_j^* \in [a_j, b_j]$

such that $4-4\lambda^q = |f'_{\lambda,p/q}(E_j^*)|(b_j-a_j)$, so $4(1-\lambda) = \nu(S_-(\lambda,p/q)) = \sum_{j=1}^{q} b_j - a_j = \sum_{j=1}^{q} \frac{4-4\lambda^q}{|f'_{\lambda,p/q}(E_j^*)|}$. Since $f_{\lambda,p/q}(E)$ is also a polynomial in λ, we have that $E_j^* \to E_j$ when $\lambda \nearrow 1$. Therefore,

$$\sum_{j=1}^{q} \frac{1}{|f'_{1,p/q}(E_j)|} = \lim_{\lambda\to 1^-} \sum_{j=1}^{q} \frac{1}{|f'_{\lambda,p/q}(E_j^*)|} = \lim_{\lambda\to 1} \frac{1-\lambda}{1-\lambda^q} = \frac{1}{q}.$$

□

Proposition 4.6. *Given a rational p/q, we have*

$$\frac{2(\sqrt{5}+1)}{q} \le \nu(S_+(1,p/q)) \le \frac{8e}{q}. \tag{4.7}$$

Proof. For simplicity, throughout this proof we will write $f(E)$ and S_+ instead of $f_{1,p/q}(E)$ and $S_+(1,p/q)$. Let $E_1,\dots,E_q$ be the zeros of f ordered increasingly and $[a_1,b_1],\dots,[a_q,b_q]$ be the q bands of $S_+ = f^{-1}([-4,4])$ ordered in a way that $a_j < E_j < b_j$ for each j.

We will first prove the upper bound. We will use an argument for the bands when $2 \le j \le q-1$ and another for when $j \in \{1,q\}$.

Suppose f is increasing in $[a_2,b_2]$ (if it is decreasing, the calculations are completely analogous) and let $E_2^- \le a_2$ and $E_2^+ \ge b_2$ be the closest minimum and maximum ponits of f, respectively. Now let $g(E) = \frac{d}{dE}\ln|f(E)|$. Since $f(E) = \prod_{j=1}^{q}(E-E_j)$, we have $g(E) = \sum_{j=1}^{q} \frac{1}{E-E_j}$. That gives us

$$g'(E) = -\sum_{j=1}^{q} \frac{1}{(E-E_j)^2} < -\frac{1}{(E-E_2)^2}.$$

Since $E_2^\pm$ are extrema points of f, we have $g(E_2^\pm)=0$, so for $E \in (E_2,E_2^+)$ we have

$$g(E) = g(E) - g(E_2^+) = -\int_E^{E_2^+} g'(t)dt > \int_E^{E_2^+} \frac{1}{(t-E_2)^2}dt = \frac{1}{E-E_2} - \frac{1}{E_2^+-E_2}.$$

And for $E \in (E_2,b_2)$ we have, since $f(b_2)=4$,

$$\ln\frac{4}{f(E)} = \ln(f(b_2)) - \ln(f(E)) = \int_E^{b_2} g(t)dt > \int_E^{b_2} \frac{1}{t-E_2} - \frac{1}{E_2^+-E_2}dt$$
$$= \ln\left(\frac{b_2-E_2}{E-E_2}\right) - \frac{b_2-E}{E_2^+-E_2} > \ln\left(\frac{b_2-E_2}{E-E_2}\right) - 1.$$

So

$$\frac{4}{f(E)} > \frac{1}{e}\cdot\frac{b_2-E_2}{E-E_2} \implies b_2-E_2 < 4e\frac{E-E_2}{f(E)} = 4e\frac{E-E_2}{f(E)-f(E_2)}.$$

Letting $E \searrow E_2$ we get $b_2 - E_2 \leq \dfrac{4e}{f'(E_2)}$.

Similarly, for $E \in (E_2^-, E_2)$ we have

$$g(E) = g(E) - g(E_2^-) = \int_{E_2^-}^{E} g'(t)dt < -\int_{E_2^-}^{E} \frac{1}{(t-E_2)^2}dt = \frac{1}{E-E_2} - \frac{1}{E_2^- - E_2}.$$

And for $E \in (a_2, E_2)$ we have, since $f(a_2) = -4$,

$$\ln\frac{4}{|f(E)|} = \ln|f(a_2)| - \ln|f(E)| = -\int_{a_2}^{E} g(t)dt > \int_{a_2}^{E} \frac{1}{E_2^- - E_2} - \frac{1}{t-E_2}dt$$
$$= \frac{E-a_2}{E_2^- - E_2} - \ln\left(\frac{|E-E_2|}{|a_2-E_2|}\right) = -\frac{E-a_2}{E_2-E_2^-} - \ln\left(\frac{E_2-E}{E_2-a_2}\right) > -1 - \ln\left(\frac{E_2-E}{E_2-a_2}\right).$$

So, since $f(E) < 0$ in this interval,

$$\frac{4}{-f(E)} > \frac{1}{e}\cdot\frac{E_2-a_2}{E_2-E} \implies E_2-a_2 < 4e\frac{E_2-E}{-f(E)} = 4e\frac{E-E_2}{f(E)-f(E_2)}.$$

Letting $E \nearrow E_2$ we get $E_2 - a_2 \leq \dfrac{4e}{f'(E_2)}$, so $b_2 - a_2 \leq \dfrac{8e}{f'(E_2)} = \dfrac{8e}{|f'(E_2)|}$, where the equality holds for f is increasing in $[a_2, b_2]$. Clearly, the same analysis can be made for each band with $2 \leq j \leq q-1$, so for such j's, $b_j - a_j \leq \dfrac{8e}{|f'(E_j)|}$.

Let's now consider the band $[a_1, b_1]$. Since we assumed f increases in $[a_2, b_2]$, we must assume f decreases in $[a_1, b_1]$. Clearly E_2^- is the minimum of f closest to b_1 and on the interval (E_1, b_1) we can perform the same calculations as for the internal bands and obtain $b_1 - E_1 \leq \dfrac{4e}{|f'(E_1)|}$. Now, since $[a_1, b_1]$ is the left external band, there is no extremum left of a_1, but we can use that $f'(E)$ is negative and increasing in $(-\infty, b_1]$. By mean value theorem, we know that thre are $E^* \in (a_1, E_1)$ and $E^{**} \in (E_1, b_1)$ such that $4 = |f(E_1) - f(a_1)| = |f'(E^*)|(E - a_1)$ and $4 = |f(b_1) - f(E_1)| = |f'(E^{**})|(b_1 - E_1)$, but since in this interval $f'(E)$ is negative and increasing, we have $|f'(E^{**})| \geq f'(E^*)$, so we get $E_1 - a_1 \leq b_1 - E_1 \leq \dfrac{4e}{|f'(E_1)|}$. A similar analysis can be carried for $[a_q, b_q]$, so that for every $1 \leq j \leq q$ we have $b_j - a_j \leq \dfrac{8e}{|f'(E_j)|}$. Thus, by Lemma 4.5

$$\nu(S_+) = \sum_{j=1}^{q} b_j - a_j \leq \sum_{j=1}^{q} \frac{8e}{|f'(E_j)|} = \frac{8e}{q}.$$

The upper bound is then proved. We have tacitly assumed that $q > 2$ in considering that $[a_2, b_2]$ is an internal band. Nevertheless, as said in the case $q = 2$ we use for the

second band the same reasoning as for $[a_1, b_1]$ and get the inequality, so there is no loss of generality in the proof.

Let's now show the lower bound.

Since $f'(E)$ is a polynomial with $q-1$ simpe real zeros, $|f'(E)|$ attains a single maximum between two consecutive zeros of f' and it is decreasing below its smallest zero and increasing above its greatest. Therefore, $|f'|$, when restricted to $[a_j, b_j]$, has only one local maximum. If we divide each band into $[a_j, E_j]$ and $[E_j, b_j]$, then $|f'|$ must be monotone in the component that does not contain the maximum point of $|f'|$ in the band, say $\tilde{E}_j$. Besides, since f' does not change sign in each band, we have

$$\int_{a_j}^{E_j} |f'(t)|dt = \left| \int_{a_j}^{E_j} f'(t)d \right| = |f(E_j) - f(a_j)| = 4,$$

and

$$\int_{E_j}^{b_j} |f'(t)|dt = \left| \int_{E_j}^{b_j} f'(t)d \right| = |f(b_j) - f(E_j)| = 4.$$

So either $E_j - a_j \geq \dfrac{4}{|f'(E_j)|}$ (if $\tilde{E}_j > E_j$) or $b_j - E_j \geq \dfrac{4}{|f'(E_j)|}$ (if $\tilde{E}_j < E_j$).

Now, as we did for the upper bound, we will divide the analysis between the internal bands and the external bands, using $[a_2, b_2]$ and $[a_1, b_1]$ as the example of each, respectively. We will also continue to assume that f is increasing in $[a_2, b_2]$, since the decreasing case is entirely analogous.

Let $l_j = \dfrac{4}{|f'(E_j)|}$. If $b_2 - E_2 \geq l_2$, define $h_a(E) = \dfrac{d}{dE} \ln (f(E) + 4)$. Let $c_1, \ldots, c_q$ be such that $c_j = [a_j, b_j] \cap f^{-1}(\{-4\})$. So $f(E) + 4 = \prod_{j=1}^{q} (E - c_j)$ and $h_a(E) = \sum_{j=1}^{q} \dfrac{1}{E - c_j}$, which gives us

$$h_a'(E) = -\sum_{j=1}^{q} \frac{1}{(E-c_j)^2} < -\frac{1}{(E-c_2)^2} = -\frac{1}{(E-a_2)^2}. \tag{4.8}$$

Therefore, if $E \in (a_2, b_2)$,

$$h_a(E) = h_a(E) - h_a(E_2^+) = -\int_E^{E_2^+} h_a'(t)dt > \int_E^{E_2^+} \frac{1}{(t-a_2)^2} dt$$
$$= \frac{1}{E - a_2} - \frac{1}{E_2^+ - a_2} > \frac{1}{E - a_2} - \frac{1}{b_2 - a_2}.$$

So $l_2^{-1} = \dfrac{f'(E_2)}{4} = \dfrac{f'(E_2)}{f(E_2) + 4} = h_a(E_2) > \dfrac{1}{E_2 - a_2} - \dfrac{1}{b_2 - a_2} = \dfrac{b_2 - E_2}{(E_2 - a_2)(b_2 - a_2)}$, which implies, using $b_2 - E_2 > l_2 > 0$,

$$(E_2 - a_2)(b_2 - a_2) - l_2(b_2 - E_2) > 0 \implies (E_2 - a_2)^2 + (E_2 - a_2)(b_2 - E_2) - l_2(b_2 - E_2) > 0$$
$$\implies (E_2 - a_2)^2 + ((E_2 - a_2) - l_2)(b_2 - E_2) > 0 \implies (E_2 - a_2)^2 + l_2(E_2 - a_2) - l_2^2 > 0.$$

Since $E_2 - a_2 > 0$, we get $E_2 - a_2 > \dfrac{\sqrt{5}-1}{2} l_2$.

In the case $E_2 - a_2 \geq l_2$, define $h_b(E) = \dfrac{d}{dE}(\ln|f(E)-4|)$. Define the sets $d_j = [a_j, b_j] \cap f^{-1}(\{4\})$ for $1 \leq j \leq q$. So $|f(E)-4| = \prod_{j=1}^{q}(E-d_j)$ and $h_b(E) = \sum_{j=1}^{q} \dfrac{1}{E-d_j}$, which gives us

$$h_b'(E) = -\sum_{j=1}^{q} \frac{1}{(E-d_j)^2} < -\frac{1}{(E-d_2)^2} = -\frac{1}{(E-b_2)^2}. \tag{4.9}$$

Therefore, if $E \in (a_2, b_2)$,

$$h_b(E) = h_b(E) - h_a(E_2^-) = \int_{E_2^-}^{E} h_b'(t)dt < -\int_{E_2^-}^{E} \frac{1}{(t-b_2)^2}dt$$
$$= \frac{1}{E-b_2} - \frac{1}{E_2^- - b_2} = \frac{1}{b_2 - E_2^-} - \frac{1}{b_2 - E} < \frac{1}{b_2 - a_2} - \frac{1}{b_2 - E}.$$

So $l_2^{-1} = \dfrac{f'(E_2)}{4} = -\dfrac{f'(E_2)}{f(E_2)-4} = -h_b(E_2) > \dfrac{1}{b_2-E_2} - \dfrac{1}{b_2-a_2} = \dfrac{E_2-a_2}{(b_2-E_2)(b_2-a_2)}$, which implies, using $E_2 - a_2 > l_2 > 0$,

$$(b_2-E_2)(b_2-a_2) - l_2(E_2-a_2) > 0 \implies (b_2-E_2)^2 + (E_2-a_2)(b_2-E_2) - l_2(E_2-a_2) > 0$$
$$\implies (b_2-E_2)^2 + ((b_2-E_2)-l_2)(E_2-a_2) > 0 \implies (E_2-a_2)^2 + l_2(E_2-a_2) - l_2^2 > 0.$$

Since $b_2 - E_2 > 0$, we get $b_2 - E_2 > \dfrac{\sqrt{5}-1}{2} l_2$. Either way, $b_2 - a_2 = (b_2 - E_2) + (E_2 - a_2) \geq \dfrac{\sqrt{5}+1}{2} l_2$.

Now for $[a_1, b_1]$, since $|f'|$ is monotonous in this band, we know that $b_1 - E_1 \geq E_1 - a_1$, so we must have $b_1 - E_1 > l_1$. Since the largest component of the band is the left one and there is a zero of f' larger than b_1, we can use h_a to get $E_1 - a_1 > \dfrac{\sqrt{5}-1}{2} l_1$. The analogous argument with h_b is used for $[a_q, b_q]$ since the largest component must be the left one and f' has a zero smaller than a_q. Since for every j we have $b_j - a_j \geq \dfrac{\sqrt{5}+1}{2} l_j = \dfrac{2(\sqrt{5}+1)}{|f'(E_j)|}$, we conclude

$$\nu(S_+) = \sum_{j=1}^{q} b_j - a_j \geq \sum_{j=1}^{q} \frac{2(\sqrt{5}+1)}{|f'(E_j)|} = \frac{2(\sqrt{5}+1)}{q}.$$

The proof of the proposition is then finally complete.

□

We can now prove the result about the measure of the spectrum:

Proof of Theorem 4.1. First consider $0 < \lambda \leq 1$ and let p_n/q_n be the continued fraction approximants of α. By Theorem 4.3 and Proposition 4.6, we have that $\lim_{n\to\infty} \nu(S_+(\lambda, p_n/q_n)) = 4(1-\lambda)$. Now let $\{g_j(\lambda,\alpha)\}_{j\in\mathbb{N}}$ be the gaps of $S_+(\lambda,\alpha)$. Then for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{j=1}^{N_\varepsilon} \nu(g_j(\lambda,\alpha)) > \nu(G(\lambda,\alpha)) - \varepsilon.$$

By Corollary 4.4.2, we have that $\nu(G(\lambda, p_n/q_n)) \geq -32N_\varepsilon\sqrt{\lambda|\alpha - p_n/q_n|} + \sum_{j=1}^{N_\varepsilon} \nu(g_j(\lambda,\alpha))$ for sufficiently large n, which gives us

$$\liminf_{n\to\infty} \nu(G(\lambda, p_n/q_n)) \geq \nu(G(\lambda,\alpha)) - \varepsilon.$$

We have $\lim_{n\to\infty} \max S_+(\lambda, p_n/q_n) = \max S_+(\lambda,\alpha)$ and $\lim_{n\to\infty} \min S_+(\lambda, p_n/q_n) = \min S_+(\lambda,\alpha)$ by Corollary 4.4.1 . So we get

$$\begin{aligned}
4(1-\lambda) &= \limsup_{n\to\infty} \nu(S_+(\lambda, p_n/q_n)) \\
&= \limsup_{n\to\infty} (\max S_+(\lambda, p_n/q_n) - \min S_+(\lambda, p_n/q_n) - \nu(G(\lambda, p_n/q_n))) \\
&\leq \limsup_{n\to\infty} \max S_+(\lambda, p_n/q_n) + \limsup_{n\to\infty} (-\max S_+(\lambda, p_n/q_n)) + \limsup_{n\to\infty} (-\nu(G(\lambda, p_n/q_n))) \\
&= \max S_+(\lambda,\alpha) - \min S_+(\lambda,\alpha) - \liminf_{n\to\infty} \nu(G(\lambda, p_n/q_n)) \\
&\leq \max S_+(\lambda,\alpha) - \min S_+(\lambda,\alpha) - \nu(G(\lambda,\alpha)) + \varepsilon = \nu(S_+(\lambda,\alpha)) + \varepsilon.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, then $\nu(S_+(\lambda,\alpha)) \geq 4(1-\lambda)$. Now, since there are at most $q_n - 1$ gaps in $S_+(\lambda, p_n/q_n)$, Corollary 4.4.2 gives us that for sufficiently large n

$$\nu(G(\lambda,\alpha)) > \nu(G(\lambda, p_n/q_n)) - 32(q_n - 1)\sqrt{\lambda|\alpha - p_n/q_n|}.$$

And Corollary 4.4.1 gives us that for sufficiently large n

$$\max S_+(\lambda,\alpha) \leq \max S_+(\lambda, p_n/q_n) + 16\sqrt{\lambda|\alpha - p_n/q_n|}.$$

$$\min S_+(\lambda,\alpha) \geq \min S_+(\lambda, p_n/q_n) - 16\sqrt{\lambda|\alpha - p_n/q_n|}.$$

Thus $\nu(S_+(\lambda,\alpha)) \leq \nu(S_+(\lambda, p_n/q_n)) + 32q_n\sqrt{\lambda|\alpha - p_n/q_n|}$. By the hypothesis on α, taking $n \to \infty$ we get $\nu(S_+(\lambda,\alpha)) \leq 4(1-\lambda)$. So for $\lambda \leq 1$ we have $\nu(\sigma(H_{\lambda,\alpha,\theta})) = \nu(S_+(\lambda,\alpha)) = 4(1-\lambda)$. Finally, if $\lambda > 1$, then $\lambda^{-1} < 1$ and by Corollary 2.3.1 we have

$$\nu(\sigma(H_{\lambda,\alpha,\theta})) = \lambda\nu(\sigma(H_{\lambda^{-1},\alpha,\theta})) = 4\lambda(1-\lambda^{-1}) = 4(\lambda - 1) = 4|1-\lambda|.$$

□

4.2 Absence of eigenvalues

Since for α in a dense G_δ of full measure we have $\nu(\sigma(H_{1,\alpha,\theta})) = 0$, we know that the absolutely continuous spectrum is empty for such α. We will now use the work of Avila in [3] to show that the set of eigenvalues of $H_{1,\alpha,\theta}$ is empty for every irrational α and all but (explicitly given) countably many values of θ, so $\sigma(H_{1,\alpha,\theta})$ is purely singular continuous for every α that satisfy the hypothesis of Theorem 4.1 and all but countably θ.

Definition 4.7. *We say that $\theta \in \mathbb{R}$ is $\alpha-$rational if there is $k \in \mathbb{Z}$ such that $2\theta + k\alpha \in \mathbb{Z}$.*

We can now state the main theorem of this section:

Theorem 4.8. *For every irrational α and every θ that is not $\alpha-$rational, $H_{1,\alpha,\theta}$ has no eigenvalues.*

Before we can prove Theorem 4.8, we need a short lemma:

Lemma 4.9. *Let $H = H_{1,\alpha,\theta}$ and assume that $Hu = Eu$ for some $E \in \mathbb{R}$ and some nonzero $u \in \ell^2$. Define $v(x) = \sum_{n=-\infty}^{\infty} u(n)e^{2\pi inx}$ and $B(x) = \begin{pmatrix} v(x) & \overline{v(x)} \\ v(x-\alpha)e^{-2\pi i\theta} & \overline{v(x-\alpha)}e^{2\pi i\theta} \end{pmatrix}$. Then $\|B(x)\| > 0$ for almost every x and $\det(B(x))$ is constant almost everywhere. Furthermore, if θ is not $\alpha-$rational, then $\det(B(x)) \neq 0$.*

Proof. Let $A(x) = \begin{pmatrix} E - 2\cos(2\pi x) & -1 \\ 1 & 0 \end{pmatrix}$. Note that

$$
\begin{aligned}
&(E - 2\cos(2\pi x))v(x) - v(x-\alpha)e^{-2\pi i\theta} \\
&= \sum_{n=-\infty}^{\infty} Eu(n)e^{2\pi inx} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(n+1)x} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(n-1)x} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx-n\alpha-\theta)} \\
&= \sum_{n=-\infty}^{\infty} (Eu(n) - u(n-1) - u(n+1))e^{2\pi inx} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx-n\alpha-\theta)} \\
&= \sum_{n=-\infty}^{\infty} 2\cos(n\alpha+\theta)u(n)e^{2\pi inx} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx-n\alpha-\theta)} \\
&= \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx+n\alpha+\theta)} + \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx-n\alpha-\theta)} - \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx-n\alpha-\theta)} \\
&= \sum_{n=-\infty}^{\infty} u(n)e^{2\pi i(nx+n\alpha+\theta)} = e^{2\pi i\theta}v(x+\alpha).
\end{aligned}
$$

So $A(x)B(x) = B(x+\alpha)D$, where $D = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}$. Therefore, the set $\{x \in \mathbb{T} \mid \|B(x)\| = 0\}$ is invariant by $x \mapsto x + \alpha$. By ergodicity of irrational rotations, it must have measure zero, since otherwise we would have $B(x) = 0$ for almost every x, contradicting $u \not\equiv 0$.

Moreover, since $\det(D) = \det(A(x)) = 1$, $\det(B(x)) = \det B(x+\alpha)$, so again by ergodicity $\det(B(x))$ is almost everywhere constant. Now assume $\det(B(x)) = 0$ almost everywhere. Then there is a measurable function $\beta : \mathbb{T} \to \mathbb{C}$ such that $\begin{pmatrix} v(x) \\ v(x-\alpha)e^{-2\pi i\theta} \end{pmatrix} = \beta(x) \begin{pmatrix} \overline{v(x)} \\ \overline{v(x-\alpha)}e^{2\pi i\theta} \end{pmatrix}$ and $|\beta(x)| = 1$ almost everywhere. Mutiplying by $A(x)$ by the left side, we obtain the equality $e^{2\pi i\theta} \begin{pmatrix} v(x+\alpha) \\ v(x)e^{-2\pi i\theta} \end{pmatrix} = e^{-2\pi i\theta}\beta(x) \begin{pmatrix} \overline{v(x+\alpha)} \\ \overline{v(x)}e^{2\pi i\theta} \end{pmatrix}$. Using the two equations, we get a useful relation for β: $e^{-4\pi i\theta}\beta(x)\overline{v(x+\alpha)} = v(x+\alpha) = \beta(x+\alpha)\overline{v(x+\alpha)} \implies e^{-4\pi i\theta}\beta(x) = \beta(x+\alpha)$. Now consider the Fourier series of β to be $\sum_{n=-\infty}^{\infty} b_n e^{2\pi inx}$. So for every $n \in \mathbb{Z}$

$$b_n e^{-4\pi i\theta} = b_n e^{2\pi in\alpha} \implies b_n(e^{2\pi i(n\alpha+2\theta)} - 1) = 0.$$

Since $\beta \not\equiv 0$, there must be $k \in \mathbb{Z}$ such that $e^{2\pi i(k\alpha+2\theta)} = 1$, so θ is $\alpha-$rational.

□

Proof of Theorem 4.8. Assume $u \in \ell^2$ is an eigenfunction of $H_{1,\alpha,\theta}$ with eigenvalue E, where θ is not $\alpha-$rational. Let $v(x)$, $A(x)$, $B(x)$ and D be as in the proof of Lemma 4.9. Since θ is not $\alpha-$rational, $B(x)$ is invertible and $A(x) = B(x+\alpha)DB(x)^{-1}$. Define $A_k(x)$ recursively by $A_1(x) = A(x)$ and $A_k(x) = A(x+(k-1)\alpha)A_{k-1}(x)$ for $k > 1$. By induction, we see that $A_k(x) = B(x+k\alpha)D^kB(x)^{-1}$. For $k = 1$ it holds by definition and for $k > 2$ we have

$$\begin{aligned} A_k(x) &= A(x+(k-1)\alpha)A_{k-1}(x) \\ &= \left(B(x+k\alpha)D^{k-1}B(x+(k-1)\alpha)^{-1}\right)\left(B(x+(k-1)\alpha)D^{k-1}B(x)^{-1}\right) \\ &= B(x+k\alpha)D^kB(x)^{-1}. \end{aligned}$$

Now we define $\Psi_k(x) = \operatorname{Tr}(A_k(x)) - 2\cos(2\pi k\theta)$. But $2\cos(2\pi k\theta) = \operatorname{Tr}(D^k) = \operatorname{Tr}(B(x)D^kB(x)^{-1})$. Since for 2×2 matrices we have $|\operatorname{Tr}(M)| \le \sqrt{2}\|M\|$ and, if M is invertible, $\|M^{-1}\| = |\det(M)|^{-1}\|M\|$, we get

$$|\Psi_k(x)| = |\operatorname{Tr}((B(x+k\alpha) - B(x))D^kB(x)^{-1})| \le \frac{2}{|\det(B(x))|}\|B(x+k\alpha) - B(x)\|\|B(x)\|.$$

Let $C_k(x) = B(x+k\alpha) - B(x)$. By construction $\|B\|_{L^2} = 2\|v\|_{L^2} = 2\|u\| < \infty$, so if k_n is a sequence of natural numbers such that $\lim_{n\to\infty}\|k_n\alpha\|_{\mathbb{T}} = 0$, then $\lim_{n\to\infty}\|C_{k_n}\|_{L^2} = 0$. Thus, by Hölder's inequality,

$$\|\Psi_k\|_{L^1} \le \frac{2}{|\det(B(x))|}\|C_k\|_{L^2}\|B\|_{L^2} \implies \liminf_{k\to\infty}\|\Psi_k\|_{L^1} = 0.$$

Now, by the definition of A_k, we see that Ψ_k is a trigonometric polynomial $\Psi_k(x) = \sum_{n=-k}^{k} \psi_{k,n} e^{2\pi i n x}$. Writing $2\cos(2\pi(x+n\alpha)) = e^{2\pi i x}e^{2\pi i n\alpha} + e^{-2\pi i x}e^{-2\pi i n\alpha}$, the coefficients $\psi_{k,\pm k}$ can be easily computed: the exponentials $e^{\pm 2\pi i k x}$ only appear if we multiply the exponentials $e^{\pm 2\pi i x}$ of all the k matrices $A(x+n\alpha)$ in the definition of A_k, so we must have $\psi_{k,\pm k} = \prod_{n=0}^{k-1} -e^{\pm 2\pi i n\alpha} = (-1)^k e^{\pm \pi i k(k-1)\alpha}$. But $\psi_{k,k} = \int_{\mathbb{T}} \Psi_k(x) e^{-2\pi i k x} d\nu(x)$, so $1 = |\psi_{k,k}| \leq \left| \int_{\mathbb{T}} \Psi_k(x) e^{-2\pi i k x} d\nu(x) \right| \leq \|\Psi_k\|_{L^1}$ for every k, a contradiction to $\liminf_{k\to\infty} \|\Psi_k\|_{L^1} = 0$.

Therefore, for irrational α and θ not $\alpha-$rational, the point spectrum of $H_{1,\alpha,\theta}$ is empty.

□

5 An overview of sharper results

We have so far studied in detail the classification of the almost Mathieu operator for full measure sets of parameters. Although the results shown up to this point are very robust, they are not the most recent. In this chapter, we will briefly present some results that allow the change of "almost every" to "every" in the previous chapters. Such results depend on fine estimates of analytic continuation of transfer matrices and dynamical systems techniques that differ much from what we have done up to this point, which is the reason why we won't explore them thoroughly.

As we have seen in the last section of chapter 3, in the supercritical case, even though we can search for optimal conditions on which the spectrum is pure point, this classification cannot hold for every α, and even for a fixed α it cannot hold for every θ. However, the situation is pretty different in the critical and subcritical cases.

For the critical case, in [7] Avila and Krikorian used the properties of quasiperiodic $SL_2(\mathbb{R})$ cocycles and the results of [12] to prove the following result:

Proposition 5.1. *Given irrational α with bounded continued fraction elements and real θ, the spectrum of $H_{1,\alpha,\theta}$ has measure zero.*

This combined with Theorems 4.1 and 4.8 gives us that for every frequency and all but possibly countably many phases, the critical almost Mathieu operator has purely singular continuous spectrum.

For the measure of the spectrum at non-critical coupling, Jitomirskaya and Krasovsky proved in [25] a kind of continuity on α for the sets $S_+(\lambda, \alpha)$ that, though not Lipschitz, is stronger than any $\zeta-$Hölder continuity for $0 < \zeta < 1$. Using this continuity they showed:

Theorem 5.2. *For irrational α and $\lambda \neq 1$, $\nu(\sigma(H_{\lambda,\alpha,\theta})) = 4|1 - \lambda|$.*

In the subcritical case we have a long way of improvement of results. In [4], Avila and Damanik attacked the problem from the "frequency side", showing the thoerem below:

Theorem 5.3. *The integrated density of states for $H_{\lambda,\alpha,\theta}$ is absolutely continuous if and only if $\lambda \neq 1$.*

Since by a result of [13] (Remark 3.5) the Lyapunov exponent vanishes at subcritical coupling, the work of Kotani at [29] yields

Theorem 5.4. *If $\lambda < 1$, then $\sigma(H_{\lambda,\alpha,\theta})$ is absolutely continuous for every α and almost every θ.*

Avila and Jitomirskaya, on the other hand, solved the problem by the "phase side": in [5] they expanded some results obtained for the almost Mathieu operator in [6][1] to a larger class of quasiperiodic Schrödinger operators and also obtained a quantitative version of Aubry duality. The result that interests us is

Theorem 5.5. *Let* $S_{\lambda,E}(x) = \begin{pmatrix} E - 2\lambda\cos(2\pi x) & -1 \\ 1 & 0 \end{pmatrix}$. *Then for* $\lambda < 1$ *and diophantine* α *the closure of the analytic conjugacy class of the cocycle* $(\alpha, S_{\lambda,E})$ *contains a constant.*

This theorem shows that $S_{\lambda,E}$ is analytically conjugated to a cocycle in the the Eliasson's perturative regime, discussed in detail in [18]. The results in Eliasson's work then allowed them to conclude

Theorem 5.6. *For* $\lambda < 1$ *and diophantine* α*, the spectrum of* $H_{\lambda,\alpha,\theta}$ *is purely absolutely continuous for every* θ*.*

In [2], dividing the analysis based on how fast the denominators of the continued fractions approximant of α grew, Avila was able to prove, at last, the following result:

Theorem 5.7. *The spectral measures of the almost Mathieu operator are absolutely continuous if and only if* $\lambda < 1$*.*

The paper actually shows only one implication, since singularity of spectral measures for $\lambda \geq 1$ are consequences of zero measure of spectrum when $\lambda = 1$ and the positivity of Lyapunov exponent when $\lambda > 1$.

With all these results, we have a pretty complete picture of how the spectrum of the almost Mathieu operator behaves with respect to its parameters. This was the work of a large number of mathematicians for more then three deacades and it contributed with new techniques for different areas of Spectral Theory and of Dynamical Systems. For example, many results stated in this text can be extented to a larger class of quasiperiodic Schrödinger operators, some even to more general ergodic Schrödinger operators. We hope to use the work we have done so far to explore problems of the spectral theory of such operators in the future.

[1] This is the paper in which they solved the decades-long Ten Martini Problem, that is, they proved that the spectrum of the almost Mathieu operator with irrational frequency is always a Cantor set.

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Appendix

APPENDIX A – Spectral Theory

Throughout this appendix, $\mathcal{H}$ will be a complex Hilbert space and $B(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. We will state some of the basic definitions and results of the Spectral Theory for bounded self-adjoint operators. For proofs and for a wider approach, the reader is referred to [31].

Definition A.1. *The* ***spectrum*** *of $A \in B(\mathcal{H})$ is the set $\sigma(A) = \{a \in \mathbb{C} | A - aI : \mathcal{H} \to \mathcal{H}$ is not bijective.$\}$. The* ***resolvent set*** *$\rho(A)$ is defined as $\rho(A) = \mathbb{C} \setminus \sigma(A)$.*

For bounded linear operators, the spectrum is always a closed subset of the closed ball with radius $\|A\|$, and therefore the spectrum is compact. If the operator is also self-adjoint, we also obtain that every element of the spectrum is real:

Proposition A.2. *If $A \in B(\mathcal{H})$ is self-adjoint, then $\sigma(A) \subset [-\|A\|, \|A\|]$.*

Given a bounded self-adjoint operator A, for continuous $f : \sigma(A) \to \mathbb{C}$ we can define $f(A) \in B(\mathcal{H})$: for polynomials $p(x) = \sum_{k=0}^{n} c_k x^k$, set $p(A) = \sum_{k=0}^{n} c_k A^k$ with $A^0 = I$. Now, since $\sigma(A)$ is a compact subset of $\mathbb{R}$, there is a sequence of polynomials (p_n) such that $\lim_{n\to\infty} \left(\sup_{x\in\sigma(A)} |p_n(x) - f(x)| \right) = 0$. Set $f(A) = \lim_{n\to\infty} p_n(A)$. The limit exists and is well-defined by the completeness of $B(\mathcal{H})$ and the properties of norm. We can extend this definition to continuous $f : \mathbb{R} \to \mathbb{C}$ considering the function $f\mathbb{1}_{\sigma(A)}$. We can now state one version of the Spectral Theorem:

Theorem A.3. *Let $A \in B(\mathcal{H})$ be self-adjoint. Then for every $\xi \in \mathcal{H}$ there is a Borel measure μ_ξ on $\sigma(A)$ such that for every continuous $f : \sigma(A) \to \mathbb{C}$*

$$\langle \xi, f(A)\xi \rangle = \int_{\sigma(A)} f(x) d\mu_\xi(x). \tag{A.1}$$

In particular, $\|\xi\|^2 = \mu_\xi(\sigma(A))$. We can, of course, extend μ_ξ to Borel subsets of the real line defining $\mu_\xi(\mathbb{R} \setminus \sigma(A)) = 0$. The measure μ_ξ is called a ***spectral measure*** *for A at ξ.*

Corollary A.3.1. *Let $A \in B(\mathcal{H})$ be self-adjoint and $a > 0$. If $\|A\psi\| < a\|\psi\|$ for some $\psi \in \mathcal{H}$, then $(-a, a) \cap \sigma(A) \neq \varnothing$.*

Proof. We will prove the contrapositive statement. Assume $(-a, a) \subset \rho(A)$, then $|x| \geq a$ for every $x \in \sigma(A)$. Thus for every $\psi \in \mathcal{H}$

$$\|A\psi\|^2 = \langle A\psi, A\psi\rangle = \langle \psi, A^2\psi\rangle = \int_{\sigma(A)} x^2 d\mu_\psi(x) \geq a^2 \mu_\psi(\sigma(A)) = a^2\|\psi\|^2.$$

□

Let's now discuss the decomposition of $\mathcal{H}$ based on spectral measures. Given a bounded self-adjoint operator A, we define the **point subspace** of $\mathcal{H}$ as the closure of the vector space spanned by the eigenvectors of A and denote it by $\mathcal{H}_p(A)$. The **continuous subspace** is then defined as $\mathcal{H}_c(A) = \mathcal{H}_p(A)^\perp$. The **point spectrum** of A is defined as $\sigma_p(A) = \sigma(A|_{\mathcal{H}_p(A)})$ and the **continuous spectrum** of A is defined as $\sigma_c(A) = \sigma(A|_{\mathcal{H}_c(A)})$. The point spectrum is the closure of the set of eigenvalues.

Proposition A.4. *Given self-adjoint $A \in B(\mathcal{H})$, then*

- *There is a countable set $\Lambda \subset \mathbb{R}$ (which can be taken as the set of eigenvalues of A), such that $\mathcal{H}_p(A) = \{\xi \in \mathcal{H} | \mu_\xi(\mathbb{R} \setminus \Lambda) = 0\}$;*
- $\mathcal{H}_c(A) = \{\xi \in \mathcal{H} | \mu_\xi(\{x\}) = 0 \ \forall x \in \mathbb{R}\}$.

We can also define the **singular subspace** as $\mathcal{H}_s(A) = \{\xi \in \mathcal{H} | \mu_\xi \perp \nu\}$ and the **absolutely continuous subspace** as $\mathcal{H}_{ac}(A) = \{\xi \in \mathcal{H} | \mu_\xi \ll \nu\}$. Both are closed subspaces. The **singular continuous subspace** is $\mathcal{H}_{sc}(A) = \mathcal{H}_s(A) \cap \mathcal{H}_c(A)$, that is, the set of vectors ξ such that μ_ξ is singular continuous with respect to Lebesgue measure. The singular continuous spectrum of A is defined as $\sigma_{sc}(A) = \sigma(A|_{\mathcal{H}_{sc}(A)})$, the **absolutely continuous spectrum** of A is defined as $\sigma_{ac}(A) = \sigma(A|_{\mathcal{H}_{ac}(A)})$, and the **singular spectrum** of A is defined as $\sigma_s(A) = \sigma(A|_{\mathcal{H}_s(A)})$.

Proposition A.5. *Given a bounded self-adjoint operator A we have*

- $\mathcal{H}_s(A) = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A)$;
- $\mathcal{H}_c(A) = \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A)$;
- $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A)$;
- $\sigma(A) = \sigma_p(A) \cup \sigma_{sc}(A) \cup \sigma_{ac}(A)$. *The union is not necessarily disjoint.*

We then say that the spectrum of A is **pure point** if $\sigma_c(A) = \varnothing$, that it is **purely singular continuous** if $\sigma_p(A) = \sigma_{ac}(A) = \varnothing$, and that it is **purely absolutely continuous** if $\sigma_s(A) = \varnothing$.

APPENDIX B – Ergodic Theory

In this appendix, we state some results of Ergodic Theory. A discussion of Theorems B.6 and B.7 can be found in [19]. Every other result can be found in [14].

Definition B.1. *Let $(X, \mathcal{A}, \mu)$ be a probability space. We say that a measurable bijection $T : X \to X$ is **ergodic** if it obeys the folowing conditions:*

- *μ is $T-$invariant, that is, $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$;*
- *if $A \in \mathcal{A}$ is such that $T^{-1}(A) = A$, then $\mu(A)\mu(X \setminus A) = 0$.*

Theorem B.2. *Let $(X, \mathcal{A}, \mu)$ be a probability space and $T : X \to X$ an ergodic transformation. If $f : X \to \mathbb{R}$ is a measurable function such that $f(T(x)) = f(x)$ for every $x \in X$, then f is $\mu-$almost everywhere constant.*

Definition B.3. *Let X be a compact metric space. An homeomorphism $T : X \to X$ is **uniquely ergodic** if there is a unique Borel probability measure μ on X such that μ is $T-$invariant. An uniquely ergodic transformation is ergodic with respect to such μ. Indeed, if there were a Borel set E such that $T^{-1}(E) = E$ and $0 < \mu(E) < 1$, then $\overline{\mu}(A) = \mu(E)^{-1}\mu(A \cap E)$ would be a different $T-$invariant Borel probability measure.*

Proposition B.4. *Let $\alpha \in \mathbb{R}$ and let $T_\alpha : \mathbb{T} \to \mathbb{T}$ be the homeomorphism $T_\alpha(x) = x + \alpha \mod 1$. Then T_α is uniquely ergodic if and only if α is irrational. In such case, the unique $T_\alpha-$invariant probability Borel measure is the Lebesgue measure on $\mathbb{T}$.*

Theorem B.5 (Denjoy-Koksma Inequality)**.** *Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function and T_α as in Proposition B.4 with irrational α. If p/q is a irreducible fraction with $q > 0$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$, then for any $\theta \in \mathbb{T}$*

$$\left|\sum_{n=0}^{q-1} g(T^n(\theta)) - q\int_{\mathbb{T}} g(\theta)d\nu(\theta)\right| \leq Var(g). \tag{B.1}$$

Theorem B.6 (Kingman's Subadditive Ergodic Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a probability space and $T : X \to X$ an ergodic transformation. If (f_n) is a sequence of integrable functions such that $f_{n+m}(x) \leq f_n(x) + f_m(T^n(x))$ for $\mu-$almost every $x \in X$ and every $n, m \in \mathbb{N}$, then for $\mu-$almost every $x \in X$ $\lim_{n\to\infty} \frac{f_n(x)}{n}$ exists and*

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \lim_{n\to\infty} \frac{1}{n}\int_X f_n(x)d\mu(x) = \inf_n \frac{1}{n}\int_X f_n(x)d\mu(x). \tag{B.2}$$

When the transformation is uniquely ergodic, we also have the following result:

Theorem B.7 (Subadditive Ergodic Theorem for Uniquely Ergodic Systems)**.** *Let X be a compact metric space and T a uniquely ergodic transformation. Let μ be the only $T-$invariant Borel probability measure. If (f_n) is a sequence of integrable functions such that $f_{n+m}(x) \leq f_n(x) + f_m(T^n(x))$ for every $x \in X$ and every $n, m \in \mathbb{N}$, then for every $x \in X$ and uniformly on X we have*

$$\limsup_{n\to\infty} \frac{f_n(x)}{n} \leq \lim_{n\to\infty} \frac{1}{n} \int_X f_n(x) d\mu(x) = \inf_n \frac{1}{n} \int_X f_n(x) d\mu(x). \tag{B.3}$$

Before we end this section, we will discuss an example of interest. Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function and T_α as in Proposition B.4 with irrational α. For $\theta \in \mathbb{T}$, define an operator H on ℓ^2 as in (C.1) with potential $V_\theta(n) = g(T_\alpha^n(\theta))$. Let $M(\cdot, \theta)$ be the transfer matrix associated with V_θ, so that $M(n+m, \theta) = M(m, T_\alpha^n \theta) M(n, \theta)$. Defining $f_n(\theta) = \ln \|M(n, \theta)\|$, we then have

Corollary B.7.1. *For every $\theta \in \mathbb{T}$,*

$$\limsup_{n\to\infty} \frac{\ln \|M(n,\theta)\|}{n} \leq \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|M(n,\theta)\| d\nu(\theta) = \inf_n \frac{1}{n} \int_{\mathbb{T}} \ln \|M(n,\theta)\| d\nu(\theta). \tag{B.4}$$

APPENDIX C – Transfer Matrices

Let $V : \mathbb{Z} \to \mathbb{R}$ be a bounded function, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\theta \in \mathbb{R}$. Let H be defined on ℓ^2 by

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n). \tag{C.1}$$

We call the function V a potential.

For fixed $E \in \mathbb{R}$ we define the **step matrix** $A : \mathbb{Z} \to SL_2(\mathbb{R})$ by $A(n) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}$ and the **transfer matrix** by $T(n) = \begin{cases} I, & \text{if } n = 0 \\ A(n-1)T(n-1), & \text{if } n > 0 \\ A(n)^{-1}T(n+1), & \text{if } n < 0 \end{cases}$. The dynamical meaning of A and T is that if $u : \mathbb{Z} \to \mathbb{C}$ is a solution of the eigenvalue equation $Hu = Eu$, then it follows from definition that $\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(n) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$ and $\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = T(n) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$, so the eigenvalue equation is entirely defined by V and its initial conditions $u(-1)$, $u(0)$.

Proposition C.1. *For fixed $E \in \mathbb{R}$, let A_1 and A_2 be the step matrices associated with potentials V_1 and V_2, respectively, and T_1 and T_2 the respective transfer matrices. Define C as the constant $C = \max\left\{\sqrt{2 + (|E| + \|V_1\|_\infty)^2}, \sqrt{2 + (|E| + \|V_2\|_\infty)^2}\right\}$. Then for $n \geq 0$*

$$\|T_1(n) - T_2(n)\| \leq nC^{n-1} \left(\max_{0 \leq m \leq n} |V_1(m) - V_2(m)| \right).$$

and for $n \leq 0$,

$$\|T_1(n) - T_2(n)\| \leq |n|C^{|n|-1} \left(\max_{n \leq m \leq 0} |V_1(m) - V_2(m)| \right).$$

Proof. First note that $\|A_1(n) - A_2(n)\| = \|A_1(n)^{-1} - A_2(n)^{-1}\| = |V_1(n) - V_2(n)|$. Furthermore, $\|A_j(n)\| = \|A_j(n)^{-1}\| = \sqrt{2 + (E - V_j(n))^2} \leq \sqrt{2 + (|E| + \|V_j\|_\infty)^2} \leq C$, so for the transfer matrix we get $\|T_j(n)\| \leq \|A_j\|_\infty^{|n|} \leq C^{|n|}$.

We now prove the statement by induction. If $n = 0$, both inequalities are immediate (and equivalent). Let $n > 0$ and assume the proposition holds for $n - 1$. Then

$$
\begin{aligned}
&\|T_1(n) - T_2(n)\| = \|A_1(n-1)T_1(n-1) - A_2(n-1)T_2(n-1)\| \\
&= \|A_1(n-1)T_1(n-1) - A_2(n-1)T_1(n-1) + A_2(n-1)T_1(n-1) - A_2(n-1)T_2(n-1)\| \\
&\leq \|A_1(n-1) - A_2(n-1)\|\|T_1(n-1)\| + \|A_2(n-1)\|\|T_1(n-1) - T_2(n-1)\| \\
&\leq |V_1(n-1) - V_2(n-1)|C^{n-1} + \sqrt{2 + (|E| + \|V_2\|_\infty)^2}(n-1)C^{n-2}\left(\max_{0\leq m\leq n-1}|V_1(m) - V_2(m)|\right) \\
&\leq C^{n-1}|V_1(n-1) - V_2(n-1)| + (n-1)C^{n-1}\left(\max_{0\leq m\leq n-1}|V_1(m) - V_2(m)|\right) \\
&\leq nC^{n-1}\left(\max_{0\leq m\leq n-1}|V_1(m) - V_2(m)|\right) \leq nC^{n-1}\left(\max_{0\leq m\leq n}|V_1(m) - V_2(m)|\right).
\end{aligned}
$$

Now let $n < 0$ and assume the proposition holds for $n + 1$. Then

$$
\begin{aligned}
&\|T_1(n) - T_2(n)\| = \|A_1(n)^{-1}T_1(n+1) - A_2(n)^{-1}T_2(n+1)\| \\
&= \|A_1(n)^{-1}T_1(n+1) - A_2(n)^{-1}T_1(n+1) + A_2(n)^{-1}T_1(n+1) - A_2(n)^{-1}T_2(n+1)\| \\
&\leq \|A_1(n)^{-1} - A_2(n)^{-1}\|\|T_1(n+1)\| + \|A_2(n)^{-1}\|\|T_1(n+1) - T_2(n+1)\| \\
&\leq |V_1(n) - V_2(n)|C^{|n+1|} + \sqrt{2 + (|E| + \|V_2\|_\infty)^2}|n+1|C^{|n+1|-1}\left(\max_{n+1\leq m\leq 0}|V_1(m) - V_2(m)|\right) \\
&\leq C^{|n+1|}|V_1(n) - V_2(n)| + (-n-1)C^{|n+1|}\left(\max_{n+1\leq m\leq 0}|V_1(m) - V_2(m)|\right) \\
&\leq -nC^{-n-1}\left(\max_{n\leq m\leq 0}|V_1(m) - V_2(m)|\right) = |n|C^{|n|-1}\left(\max_{0\leq m\leq n}|V_1(m) - V_2(m)|\right).
\end{aligned}
$$

□

APPENDIX D – Continued Fractions

In this appendix we will state the main results of the representation of real numbers by continued fractions. The details and proofs can be found at [28].

Definition D.1. *A **continued fraction** is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \tag{D.1}$$

if it is infinite, or of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}, \tag{D.2}$$

if it is finite.

We represent an expression of the form (D.1) by $[a_0; a_1, a_2, a_3, \ldots]$ and an expression of the form (D.2) by $[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$. We call a_j the **elements** of the continued fractions. A priori, the elements can be any kind of number (or even variables), but for the theory of rational approximants of real numbers, we always take $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$ for $j \geq 1$. Since $[a_0; a_1, a_2, \ldots, a_{n-1}, 1] = [a_0; a_1, a_2, \ldots, a_{n-1} + 1]$, we also use the convention that if a finite continued fraction has more than one element, then its last element is always greater than 1.

We define the k^{th} order approximant of a continued fraction as the finite continued fraction represented by $s_k = [a_0; a_1, a_2, \ldots, a_k]$. Of course, the approximants of a finite continued fraction with $n + 1$ elements are only defined for $k \leq n$ and the n^{th} order approximant equals the continued fraction itself. We say that an infinite fraction converges if the sequence $(s_k)_{k\geq 0}$ converges, and we say that $\alpha = \lim_{k\to\infty} s_k$ is the value of the continued fraction. The following result tells us how the elements of a continued fraction can be used to determine its convergence:

Theorem D.2. *A continued fraction $[a_0; a_1, a_2, a_3, \ldots]$ converges if and only if $\sum_{j=0}^{\infty} a_j$ diverges.*

Since in our convention the elements of a continued fraction are natural (except possibly for a_0), an infinite continued fraction always converges. We can also determine the k^{th} order approximant recursively:

Theorem D.3. *Given a continued fraction $[a_0; a_1, a_2, a_3, \ldots]$, the k^{th} order approximant is a rational $s_k = \frac{p_k}{q_k}$ where $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$, $q_1 = a_1$ and, for $k \geq 2$,*

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2}, \\ q_k = a_k q_{k-1} + q_{k-2}. \end{cases}$$

The fraction $\frac{p_k}{q_k}$ is always irreducible.

Not only it can be easily shown that $q_k \to \infty$ when $k \to \infty$, but we also have a lower bound on the growth of p_k:

Proposition D.4. *For $k \geq 2$, we have $q_k \geq 2^{(k-1)/2}$.*

The same way we have decimal approximations (or in a base b) for real numbers, we can also use continued fractions to approximate real numbers:

Theorem D.5. *For every $\alpha \in \mathbb{R}$ there is a unique continued fraction representation. The continued fraction is finite if and only if $\alpha \in \mathbb{Q}$.*

The elements of the continued fraction representation are recursively defined: set $a_0 = \lfloor \alpha \rfloor$ and define the first remainder r_1 by $r_1^{-1} = \alpha - a_0$; then for $j \geq 1$, set $a_j = \lfloor r_j \rfloor$; if $r_j \in \mathbb{Z}$ the process stops; otherwise define the next remainder by $r_{j+1}^{-1} = r_j - a_j$.

The continued fraction representation has a great advantage over the decimal representation because the approximants p_k/q_k are the best approximations for a real number in the following sense:

Theorem D.6. *Given $\alpha \in \mathbb{R}$, let p_k/q_k be the k^{th} order approximant of its continued fraction representation. If $a, b \in \mathbb{Z}$ with $0 < b \leq q_k$ and $a/b \neq p_k/q_k$ then*

$$|q_k \alpha - p_k| \leq |b\alpha - a|.$$

Furthermore, the only case where equality holds is if $\alpha = a_0 + \frac{1}{2}$, when for the zeroth approximant we have $|1 \cdot \alpha - a_0| = |1 \cdot \alpha - (a_0 + 1)|$.

It is natural to ask oneself how fast the continued fraction approximants of an irrational α converge. In a first approach, we get the inequalities $\frac{1}{q_n(q_{n+1} + q_n)} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$, so the convergence is at least as fast as the growth of the denominators squared. If the rate of convergence is greater than that depends on the (un)boundedness of the continued fraction elements:

Theorem D.7. *Let α be an irrational a_j its continued fraction elements. If there is $M > 0$ such that $|a_j| < M$ for every $j \geq 0$, then there exists $c > 0$ such that no pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ satisfies the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$. On the other hand, if the sequence of elements is unbounded, for any $c > 0$ there are infinite pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$.*

One may then wonder which is more "common" for a irrational number: whether its continued fraction elements form an unbounded sequence or not. The answer is given by the following theorem:

Theorem D.8. *The set of $\alpha \in \mathbb{R}$ for which the continued fraction elements are bounded has Lebesgue measure zero.*

It is then known that almost every real number has a better rate of convergence than the square of its continued fraction approximants' denominators. Nevertheless, almost every real number admits only a polynomial rate of convergence.

Definition D.9. *We say that $\alpha \in \mathbb{R}$ is **diophantine** if there exist $c > 0$ and $n \in \mathbb{N}$ such that $|q\alpha - p| \geq cq^{-n}$ for every pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Clearly, a rational number cannot be diophantine, so we could also use the strict inequality. An irrational number which is not diophantine is called a **Liouville number**.*

We now state the last results of this chapter. For the reader's convenience, we outline a proof of them since they are not stated in [28].

Proposition D.10. *If α is diophantine and $\frac{p_k}{q_k}$ are its continued fraction approximants, then there are $c > 0$ and $n \in \mathbb{N}$ such that*

$$q_{k+1} < c^{-1} q_k^n \tag{D.3}$$

for every $k \geq 0$

Proof. Since $\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}$ there are $c > 0$ and $n \in \mathbb{N}$ such that

$$\frac{1}{q_k q_{k+1}} > \left|\alpha - \frac{p_k}{q_k}\right| = \frac{1}{q_k} |q_k \alpha - p_k| > c q_k^{-n-1} \implies q_{k+1} < c^{-1} q_k^n.$$

□

Theorem D.11. *The set of Liouville numbers has Lebesgue measure zero.*

Proof. Let $\mathbb{L}$ be the set of Liouville numbers. Then $\alpha \in \mathbb{L}$ if and only if it is irrational and for every real $c > 0$ and natural n there are $p, q \in \mathbb{Z}$ with $q > 0$ such that $|q\alpha - p| \leq cq^{-n}$. This is equivalent to for every natural $n > 1$ there are $p, q \in \mathbb{Z}$ with $q > 1$ such that

$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$. Define for each n the open set $U_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right) \setminus \left\{\frac{p}{q}\right\}$, so that $\mathbb{L} = \bigcap_{n=2}^{\infty} U_n$. Let k be an integer and $U_{n,k} = U_n \cap [k, k+1)$, so that $U_{n,k} \subset \bigcup_{q=2}^{\infty} \bigcup_{p=kq}^{(k+1)q} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)$. We then have

$$\nu\left(\bigcup_{q=2}^{\infty} \bigcup_{p=kq}^{(k+1)q} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)\right) \leq \sum_{q=2}^{\infty} \sum_{p=kq}^{(k+1)q} \nu\left(\left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)\right) = \sum_{q=2}^{\infty} \sum_{p=kq}^{(k+1)q} \frac{2}{q^n}$$
$$= \sum_{q=2}^{\infty} \frac{2}{q^{n-1}} \leq \int_1^{\infty} \frac{2}{q^{n-1}} dq = \frac{2}{(n-2)q^{n-2}} \leq \frac{2}{n-2},$$

whence $\bigcap_{n=2}^{\infty} U_{n,k}$ has measure zero. In the second inequality, we tacitly assumed $n > 2$. This does not jeopardize the argument since $U_{n+1,k} \subset U_{n,k}$ for every n. Using that

$$\mathbb{L} = \bigcup_{k\in\mathbb{Z}} \mathbb{L} \cap [k, k+1) = \bigcup_{k\in\mathbb{Z}} \left(\left(\bigcap_{n=2}^{\infty} U_n\right) \cap [k, k+1)\right)$$
$$= \bigcup_{k\in\mathbb{Z}} \left(\bigcap_{n=2}^{\infty} U_n \cap [k, k+1)\right) = \bigcup_{k\in\mathbb{Z}} \bigcap_{n=2}^{\infty} U_{n,k},$$

we see that $\mathbb{L}$ is the countable union of sets of zero Lebesgue measure, so $\nu(\mathbb{L}) = 0$. □

Since the rationals also have measure zero, the diophantine numbers form a set of full Lebesgue measure.